Ordinary differential equations (MA5009)

Solution key - Surprise Quiz 1

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1. Define an auxiliary function

$$R(t) = \exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right)\int_a^t \chi(s)\varphi(s)\,\mathrm{d}s$$

This and an application of product rules show that

$$\begin{aligned} R'(t) &= \exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right)\chi(t)\varphi(t) - \left(\int_a^t \chi(s)\varphi(s)\,\mathrm{d}s\right)\exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right)\chi(t) \\ &= \chi(t)\exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right)\left(\varphi(t) - \left(\int_a^t \chi(s)\varphi(s)\,\mathrm{d}s\right)\right) \\ &\leq \chi(t)\psi(t)\exp\left(-\int_a^t \chi(s)\,\mathrm{d}s\right). \end{aligned}$$

Observe that $R(a) = 0$. This and the last displayed inequality imply

$$R(t) \leq \int_a^t \chi(s)\psi(s)\exp\left(-\int_a^s \chi(r)\,\mathrm{d}r\right)\mathrm{d}s.$$

This show

$$\begin{aligned} \int_a^t \chi(s)\varphi(s) &\leq \exp\left(\int_a^t \chi(s)\,\mathrm{d}s\right)\int_a^t \chi(s)\psi(s)\exp\left(-\int_a^s \chi(r)\,\mathrm{d}r\right)\mathrm{d}s \\ &= \int_a^t \chi(s)\psi(s)\left(\exp\left(-\int_a^s \chi(r)\,\mathrm{d}r\right)\exp\left(\int_a^t \chi(s)\,\mathrm{d}s\right)\right)\mathrm{d}s \\ &= \int_a^t \chi(s)\psi(s)\left(\exp\left(\int_s^t \chi(s)\,\mathrm{d}s\right)\right)\mathrm{d}s. \end{aligned}$$

This and the hypotheses in the problem show that

$$\varphi(t) \leq \psi(t) + \int_a^t \chi(s)\psi(s)\left(\exp\left(\int_s^t \chi(s)\,\mathrm{d}s\right)\right)\mathrm{d}s.$$