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Theorem (Sturm comparison theorem). Let φ_1 and φ_2 are two solutions of the problems

$$y'' + q_1(x)y = 0 \quad (1)$$
$$y'' + q_2(x)y = 0 \quad (2)$$

with $q_1(x) \le q_2(x)$ $\forall x \in I$. Then between any two consecutive zeros x_1 and x_2 of φ_1 (soln of (1)), there exists at least a zero of φ_2; unless $\varphi_1 \equiv \varphi_2$.

Proof. Assume that x_1 and x_2 are two consecutive zeros of φ_1; wlog $x_1 < x_2$ and $\varphi_1 > 0$. Recall from previous arguments that

$$\varphi_1'(x_1) > 0 \quad \text{and} \quad \varphi_1'(x_2) < 0$$

(see fig 1).

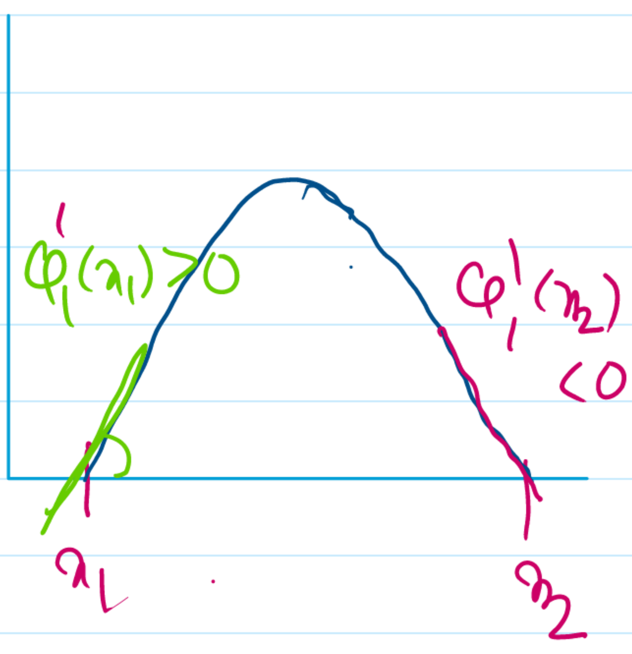

Fig. 1

Assume, if possible, that φ_2 does not have a zero in (x_1, x_2); wlog $\varphi_2 > 0$ on $[x_1, x_2]$

$$\varphi_2\varphi_1'' + q_1(x)\varphi_1\varphi_2 = 0 \quad (\text{from } (1)) \quad (3)$$
$$\varphi_1\varphi_2'' + q_2(x)\varphi_2\varphi_1 = 0 \quad (\text{from } (2)) \quad (4)$$

$$(3) - (4) \Rightarrow \varphi_2\varphi_1'' - \varphi_1\varphi_2'' + \varphi_1\varphi_2(q_1 - q_2) = 0$$

$$\Rightarrow (\varphi_2\varphi_1' - \varphi_1\varphi_2')' + \varphi_1\varphi_2(q_1 - q_2) = 0 \quad (5)$$

Integrate (5) on (x_1, x_2)

$$\varphi_2(x_2)\varphi_1'(x_2) - \varphi_1(x_2)\varphi_2'(x_2) - \varphi_2(x_1)\varphi_1'(x_1) + \varphi_1(x_1)\varphi_2'(x_1)$$

(with $\varphi_1(x_2) = 0$ and $\varphi_1(x_1) = 0$)

$$= \int_{x_1}^{x_2} \varphi_1\varphi_2(q_2 - q_1)\,dx$$

$$0 > \varphi_2(x_2)\varphi_1'(x_2) - \varphi_2(x_1)\varphi_1'(x_1) = \int_{x_1}^{x_2} \varphi_1\varphi_2(q_2 - q_1)\,dx \ge 0$$

$>0 \quad <0 \quad >0 \quad >0$ $\quad [\varphi_1, \varphi_2 > 0, \; q_2 \ge q_1]$

This contradiction shows that φ_2 should've a zero in (x_1, x_2).

Applications (Bessel's equation)

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (6)$$

Exc. use the substitution $y = v/\sqrt{x}$ to transform (6) into the form

$$v'' + \left(1 + \frac{1-4\nu^2}{4x^2}\right)v = 0 \quad (x > 0) \quad (7)$$

Case 1. $0 < \nu \le \frac{1}{2}$

$$1 + \frac{1-4\nu^2}{4x^2} \ge 1$$

$$v'' + 1 \cdot v = 0 \qquad q_2 = 1 + \frac{1-4\nu^2}{4x^2} \quad (8)$$
$$q_1 = 1$$

By SCT, between any two zeros of (8), there should exists at the least one zero of φ_2.

Note that $\varphi_1 = \sin x$ is a soln of (8) and consecutive zeros of φ_1 are $((n-1)\pi, n\pi)$ with $n \in \mathbb{Z}$. Therefore, between $(n-1)\pi$ and $n\pi$, there should exists a zero of Bessel's soln.

Exc. What if $\nu > \frac{1}{2}$.