October 7 (Monday) - Lecture 1

07 October 2024 11:08

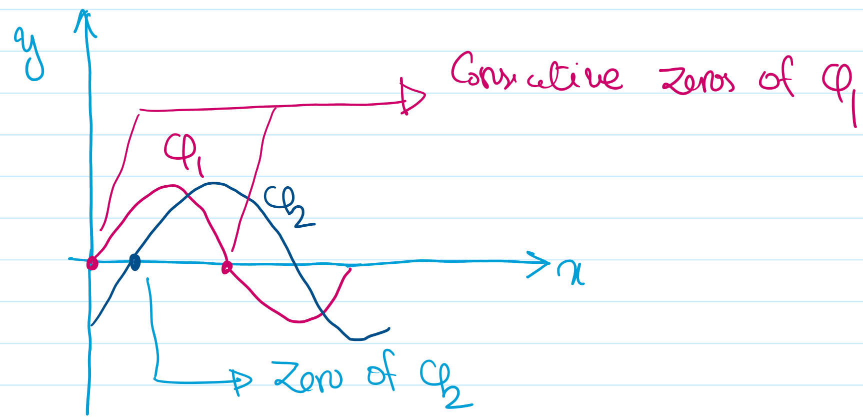
Sturm Theory (Unit 1)

- Sturm Separation Theorem
- Sturm Comparison Theorem

Context: $L[y] = 0$; where $L[y] = y''(x) + a(x)y'(x) + b(x)y(x)$

- Let φ_1 and φ_2 are two linearly ind. solutions of $L[y]=0$

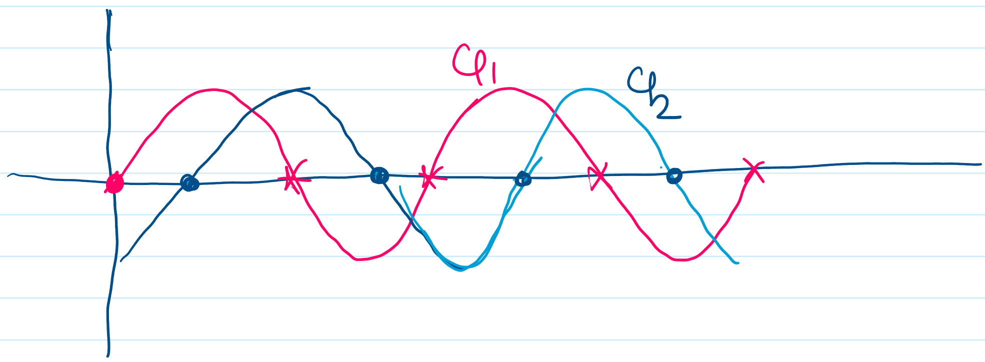
SST: Zeros of φ_1 are "separated" by zeros of φ_2 and v.v.



Example: $y'' + y = 0$

CE: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

Linearly ind. solutions are $\varphi(x) = \sin x$ and $\varphi_2(x) = \cos x$.



Theorem (Sturm separation) Suppose that φ_1 and φ_2 are two linearly independent solutions of $L[y] = 0$. Then, in between any two consecutive zeros of φ_1, $\exists$ exactly one zero of φ_2 and vice versa.

Proof. Assume that x_1 and x_2 are two zeros of φ_1

and wlog assume that $x_1 < x_2$. $(\varphi_1(x_1) = 0 = \varphi_1(x_2))$

$$W(\varphi_1, \varphi_2; x_1) = \det \begin{bmatrix} \varphi_1(x_1) & \varphi_2(x_1) \\ \varphi_1'(x_1) & \varphi_2'(x_1) \end{bmatrix} = \underbrace{\cancel{\varphi_1(x_1)}}_{=0}\varphi_2'(x_1) - \varphi_2(x_1)\varphi_1'(x_1)$$

$$= -\varphi_2(x_1)\varphi_1'(x_1) \Rightarrow \varphi_2(x_1) \neq 0$$

(o.w, φ_1 and φ_2 become LD)

Similarly $\varphi_2(x_2) \neq 0$.

If possible assume that $\varphi_2(x) > 0 \ \forall x \in (x_1, x_2)$ and $\varphi_2(x_1) \neq 0$, $\varphi_2(x_2) \neq 0$.

We can define the f^n $u = \dfrac{\varphi_1}{\varphi_2}$ on $[x_1, x_2]$

$$u(x_1) = 0, \quad u(x_2) = 0$$

By, Rolle's Thm, $\exists\ \xi \in (x_1, x_2)$ such that

$$u'(\xi) = 0$$

But

$$u'(\xi) = \frac{\varphi_2(\xi)\varphi_1'(\xi) - \varphi_2'(\xi)\varphi_1(\xi)}{\varphi_2^2(\xi)}$$

$$= -W(\varphi_1, \varphi_2; \xi) / \varphi_2^2(\xi) = 0$$

$$\Rightarrow -W(\varphi_1, \varphi_2; \xi) = 0$$

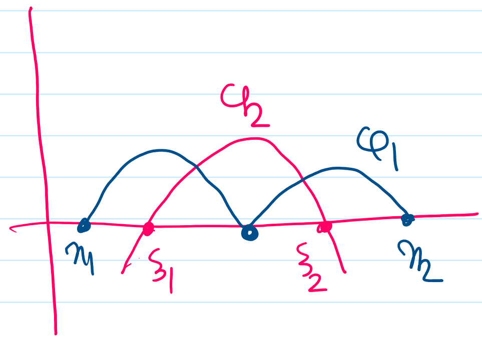
This is not possible as φ_1 and φ_2 are LI.

$\Rightarrow \exists$ at least one zero of φ_2 in (x_1, x_2).

Suppose there are two zeros of φ_2 b/w x_1 and x_2.

then, same arguments reveal that

then, same arguments reveal that $\exists$ a zero $\tilde{x}$ of φ_1 between zeros of φ_2 (ξ_1 & ξ_2). This violates the consecutiveness of zeros of φ_1 and φ_2.



Theorem. Let $q(x) \leq 0$ in the interval (x_1, x_2). Then any nontrivial solution φ of the eq^n

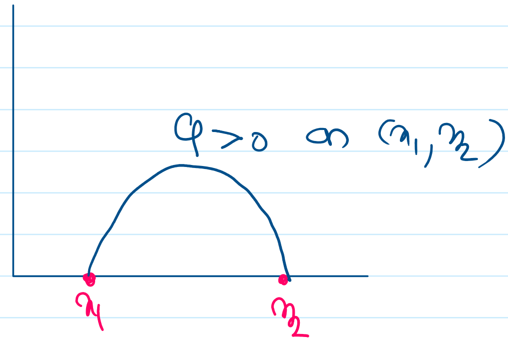
$$y'' + q(x)y = 0$$

has at the most one zero in (x_1, x_2).

Proof. Suppose that x_1 and x_2 $(x_1 < x_2)$ are two consecutive zeros of φ. and wlog $\varphi > 0$ on (x_1, x_2).

Claim: $\varphi'(x_1) > 0$

If $\varphi'(x_1) = 0$, then the unique solution is $\varphi \equiv 0$, which violates the hypothesis that φ is a non trivial solution.



$\varphi'(x_1) < 0$, Then there exists a small enough nbd around x_1 such that $\varphi(x) < 0$ in this nbd. This is also not possible $\varphi > 0$ on (x_1, x_2).

therefore $\varphi'(x_1) > 0$.

Similarly $\varphi'(x_2) < 0$.

We've

$$\varphi''(x) = -q(x)\varphi(x) \qquad \forall\, x \in (x_1, x_2)$$
$$\geq 0$$

$\Rightarrow$ φ' is a monotonically increasing function.

$$x_2 > x_1$$

$$0 > \varphi'(x_2) \geq \varphi'(x_1) > 0.$$

this is a contradiction. Therefore φ cannot have two zeros.