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Prüfer Substitution (Cont^d)

Theorem. Let $p(x)>0$ and $p \in \mathcal{C}^1[a,b]$, $q \in \mathcal{C}[a,b]$. Then the only solution of

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = 0 \qquad ①$$

which has infinitely many zeros on $[a,b]$ is the trivial solution

[i.e., any non-trivial solution can admit only finite # of zeros]

Proof. Suppose that φ solves ① and φ has infinitely many number of zeros $(x_n)_{n\geq 1}$ on $[a,b]$.

By Bolzano-Weirstrass Theorem, $\exists$ a convergent subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$. Such that $x_{n_k} \longrightarrow c \in [a,b]$.

$$\varphi(c) = \varphi\left(\lim_{k\to\infty} x_{n_k}\right) = \lim_{k\to\infty} \varphi(x_{n_k}) = 0 \quad \text{(By Cont. of } \varphi\text{)}.$$

$$\varphi'(c) = \lim_{k\to\infty} \frac{\varphi(x_{n_k}) - \varphi(c)}{x_{n_k} - c} = \lim_{k\to\infty} \frac{0}{x_{n_k} - c} = 0$$

Since $\varphi(c) = 0 = \varphi'(c)$, the unique solution is $\varphi \equiv 0$.

Theorem. Let $p_1, p_2 \in \mathcal{C}^1[a,b]$ and $0 < p_2(x) \leq p_1(x)$; $q_1, q_2 \in \mathcal{C}[a,b]$ and $q_2(x) \leq q_1(x)$ on $[a,b]$. Let φ_1 and φ_2 be nontrivial solutions of

$$-\frac{d}{dx}\left(p_1\frac{dy}{dx}\right) + q_1 y = 0 \quad \text{and}$$

$$-\frac{d}{dx}\left(p_2\frac{dy}{dx}\right) + q_2 y = 0.$$

If $\theta_2(a) \geq \theta_1(a)$, then $\theta_2(x) \geq \theta_1(x)$.

Proof. Prüfer equations are

$$\frac{d\theta_1}{dx} = \frac{1}{p_1}\cos^2\theta_1 - q_1\sin^2\theta_1 = F_1(x, \theta_1(x))$$

$$\frac{d\theta_2}{dx} = \frac{1}{p_2}\cos^2\theta_2 - q_2\sin^2\theta_2 = F_2(x, \theta_2(x))$$

Given that $p_1 \geq p_2 \Rightarrow \frac{1}{p_1} \leq \frac{1}{p_2}$

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$q_1 \geq q_2 \Rightarrow -q_1 \leq -q_2$

$\Rightarrow F_1(x, \theta(x)) \leq F_2(x, \theta(x))$

$$\frac{d}{dx}(\theta_2 - \theta_1) = F_2(x, \theta_2(x)) - F_1(x, \theta_1(x))$$

$$\geq F_1(x, \theta_2(x)) - F_1(x, \theta_1(x)).$$

$$\Rightarrow \frac{d}{dx}(\theta_1 - \theta_2) \leq F_1(x, \theta_1(x)) - F_1(x, \theta_2(x))$$

$$\leq K_1 |\theta_1(x) - \theta_2(x)| \quad (K_1 - \text{Lip Constant of } F_1)$$

If possible, assume that $\theta_1 > \theta_2$; $\xi = \theta_1 - \theta_2 > 0$.

$$\Rightarrow \frac{d\xi}{dx} \leq K_1 \xi \quad \Rightarrow \quad \xi(x) \leq \xi(a) e^{K_1(x-a)}$$

$$= (\theta_1(a) - \theta_2(a)) e^{K_1(x-a)}$$

$$\leq 0 \quad (\because \theta_1(a) \leq \theta_2(a))$$

$$\Rightarrow \theta_1(x) \leq \theta_2(x)$$

The contradiction implies $\theta_1(x) \leq \theta_2(x)$.

Corollary. If $q_2(x) < q_1(x)$ $\forall x \in (a, b]$, then $\theta_2(x) > \theta_1(x)$ on $(a, b]$.

Proof- If possible, assume that for some $c > a$, it holds

$$\theta_2(x) = \theta_1(x) \quad \forall x \in [a, c].$$

The Prüfer equations become

$$0 = \frac{d}{dx}(\theta_1 - \theta_2) = \underbrace{\cos^2\theta\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}_{\leq 0} + \underbrace{\sin^2\theta\,(q_2 - q_1)}_{< 0}$$

on (a, c).

Since $\left(1/p_1 - 1/p_2\right)\cos^2\theta \leq 0$, $\sin^2\theta\,(q_2 - q_1) < 0$, the terms should be individually zero.

$$\Rightarrow \cos^2\theta\left(\frac{1}{p_1} - \frac{1}{p_2}\right) = 0 \quad \text{and} \quad \sin^2\theta\,(q_1 - q_2) = 0.$$

$$\Rightarrow \sin\theta = 0, \quad p_1 = p_2$$

$$\Rightarrow \theta = 0 \pmod{\pi}, \quad p_1 = p_2$$

then Prüfer equations becomes,

$$0 = \frac{1}{p_1} \quad , \quad 0 = \frac{1}{p_2} \quad \text{on} \quad (a, c]$$

which is not possible.

This shows that there does not exist any $c > a$ such that

$\theta_1(x) = \theta_2(x)$ on $(a, c]$.

Choose $c = a + \frac{1}{n}$. Then $\exists$ x_n such that $\theta_2(x_n) > \theta_1(x_n)$.

As $n \to \infty$, $x_n \to a$.

Also note that

$$\theta_1(x) - \theta_2(x) \leq (\theta_1(x_n) - \theta_2(x_n)) \exp(K(x - x_n)) \quad \forall x > x_n$$

$$\Rightarrow \theta_2(x) - \theta_1(x) \geq \underbrace{(\theta_2(x_n) - \theta_1(x_n))}_{>0} \underbrace{\exp(K(x - x_n))}_{>0}$$

$$> 0$$

$$\Rightarrow \quad \theta_2(x) > \theta_1(x) \quad \forall \ x \geq x_n$$

Since as $n \to \infty$, $x_n \to a$, it holds $\theta_2(x) > \theta_1(x)$

$\forall x \in (a, b]$

Theorem (Oscillation). Let φ_1 and φ_2 be two non-trivial solutions of

$$L_1(y) = -\frac{d}{dx}(p_1 y') + q_1 y = 0$$

$$L_2(y) = -\frac{d}{dx}(p_2 y') + q_2 y = 0$$

with $0 < p_2(x) \leq p_1(x)$ and $q_2(x) \leq q_1(x)$ $\forall x \in [a, b]$. Then φ_2 has a zero between any pair of consecutive zeros of φ_1.

Proof. Let $x_1 < x_2$ be two consecutive zeros of φ_1.

$\varphi_1(x_1) = 0$ and $\varphi_1(x_2) = 0$

$\Rightarrow \theta_1(x_1) = k\pi$ and $\theta_1(x_2) = (k+1)\pi$ for some $k \in \mathbb{Z}$.

φ_2 is undisturbed by a translation of θ_2 by $2n\pi$, $n \in \mathbb{Z}$.

If necessary, with a translation,

$$\theta_2(x_1) - \theta_1(x_1) < \pi$$

$$\Rightarrow \theta_2(x_1) < (k+1)\pi$$

From the prs. thm, $\theta_2(x_2) \geq \theta_1(x_2) = (k+1)\pi$

$$\Rightarrow \quad \theta_2(x_1) < (k+1)\pi$$
$$\theta_2(x_2) \geq (k+1)\pi$$

$\therefore \exists \; x^* \in [x_1, x_2]$ such that $\theta_2(x^*) = (k+1)\pi$,

$\Rightarrow \varphi_2(x^*) = 0$.

Corollary: If $q_2 < q_1$, then $x^* \in [x_1, x_2)$.