Prüfer substitution

- Locate the zeros of a solution of the self adjoint operator

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = 0 \qquad p(x) > 0$$

Introduce the variables: $r = r(x)$ and $\theta = \theta(x)$

$$py' = r(x)\cos\theta(x) \quad \text{and} \quad y = r(x)\sin\theta(x)$$

$$(py')^2 + y^2 = r^2 \qquad \theta = \arctan\left(y/py'\right)$$

$r(x)$ – amplitude and $\theta(x)$ – phase variable

Remark. Suppose that for some $\hat{x}$, $r(\hat{x}) = 0 \Rightarrow y(\hat{x}) = 0,\ y'(\hat{x}) = 0$.
$\Rightarrow y \equiv 0$ is the only solution

Idea: Formulate an equivalent system on r and θ.

$$2r\frac{dr}{dx} = 2py'\frac{d}{dx}(py') + 2y\frac{dy}{dx}$$

$$\Rightarrow r\frac{dr}{dx} = py'\,qy + y\frac{dy}{dx} = r\cos\theta \times q \times r\sin\theta + r\sin\theta \times \frac{r\cos\theta}{p}$$

$$= \left(\frac{1}{p} + q\right) r^2 \sin\theta\cos\theta$$

$$\Rightarrow \quad \frac{dr}{dx} = \left[\frac{1}{p} + q\right] r\sin\theta\cos\theta$$

$$\arctan\left(\frac{y}{py'}\right) = \theta \Rightarrow \tan\theta = y/py'$$

$$\sec^2\theta\,\frac{d\theta}{dx} = \frac{(py')y' - y(py')'}{(py')^2} = \frac{(py')y' - y(qy)}{(py')^2}$$

$$= \frac{r\cos\theta \times \frac{r\cos\theta}{p} - r\sin\theta \times q \times r\sin\theta}{r^2\cos^2\theta}$$

$$= \frac{1}{p} - q \tan^2\theta$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{\cos^2\theta}{p} - q \sin^2\theta$$

Prüfer system

$$\frac{dr}{dx} = \left(\frac{1}{p} + q\right) r \sin\theta \cos\theta$$

$$\frac{d\theta}{dx} = \frac{\cos^2\theta}{p} - q \sin^2\theta$$

- r and θ are decoupled in the Prüfer system

$$\frac{d\theta}{dx} = f(x, \theta(x)) = \frac{1}{p}\cos^2\theta - q \sin^2\theta \qquad p \in \mathcal{C}^1[a,b] \text{ and } q \in \mathcal{C}[a,b]$$

$$\frac{\partial f}{\partial \theta} = \frac{-1}{p} 2\cos\theta\sin\theta - 2q \sin\theta\cos\theta = -\left(1/p + q\right)\sin 2\theta$$

Note that $\displaystyle \sup_{x\in[a,b]} \left|\frac{\partial f}{\partial \theta}\right| \leq \sup_{x\in[a,b]} \frac{1}{|p(x)|} + \sup_{x\in[a,b]} |q(x)|$

Therefore $f(x,\theta)$ is a Lipschitz continuous function with the Lipschitz constant $\displaystyle K_f = \sup_{x\in[a,b]} \left|\frac{\partial f}{\partial \theta}\right|$.

Therefore, if $\theta(a) = \theta_a$ is known, then by Picard's theorem, there exists a unique solution $\theta(x)$. Once $\theta(x)$ is known, $r(x)$ is given by

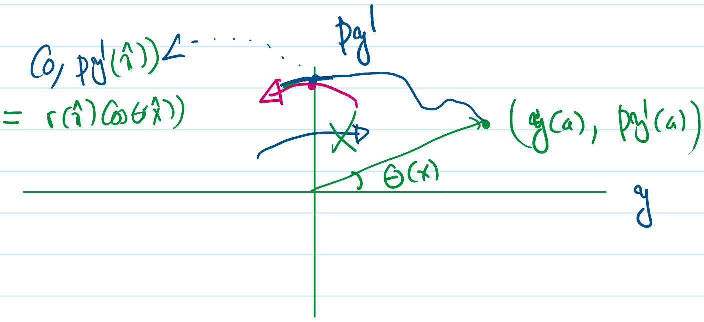
$$r(x) = r(a) \exp\left\{ \int_a^x \left(\frac{1}{p(s)} + q(s)\right) \frac{\sin[2\theta(s)]}{2} ds \right\}$$

- A change in $r(a)$ would only change $r(x)$ by a corresponding scaling factor.
- Note that $y(x) = r(x)\sin\theta(x)$. Therefore $y(x) = 0$ iff $\theta(x) = \pm n\pi$, $n \in \mathbb{N}$.

Suppose that $\theta(\hat{x}) = \pm n\pi$, then $|\cos\theta(\hat{x})| = 1$

$$\frac{d\theta(\hat{x})}{dx} = \frac{1}{p(\hat{x})} > 0$$

This means at an $\hat{x}$ for which $\theta(\hat{x}) = \pm n\pi$, $n \in \mathbb{N}$, the function $\theta(x)$ is monotonically increasing in a small neighbourhood around $\hat{x}$.



(py', y)

$x \to r \sin\theta(x) = y$
$r \cos\theta(x) = py'$

a (r(a) sin θ(a),
r(a) cos θ(a))

That is the (y, py') as a parametric curve of x cross the py' coordinate only counterclockwise