Ordinary differential equations (MA5009)

Test 1 (Answer Key)

August–November Semester 2024

Department of Mathematics, Indian Institute of Technology Palakkad

06 September, 2024 (Friday)

Maximum score: 15 *Duration: 50 minutes (8.00 - 8.50 AM)*

Instructions

1. This question sheet has **two** pages and **seven** number of questions in **three** sections. Make sure that your question sheet contains both the pages. If not, ask the invigilator to give you another question sheet with two pages.
2. The instructions for answering the questions can be found in each section. No doubts or explanations are allowed in the examination hall.
3. Any inequality or result from the course content can be assumed. In this case, the result can be used without stating. If an external result is used, a statement needs to be provided.

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1 True or false questions

This section contains four questions. Each question has one point. Answer whether the statement is true or false. No justification is required. Incorrect answers ***does not*** *have negative score.*

1. Suppose f is a continuous function on the rectangle

$$R = \{(x, y) : |x - x_0| \leq a, |y - y_0| \leq b\}.$$

Then, the initial value problem $\mathrm{d}y/\mathrm{d}x = f(x, y)$ with the initial condition $y(x_0) = y_0$ has a solution in a small enough neighbourhood of x_0.

Solution. True

2. If $M, M, \partial_y M, \partial_x N$ are continuous and $\partial_y M = \partial_x N$ for all (x, y) in a rectangular region R, then the differential $M\,\mathrm{d}x + N\mathrm{d}y$ is exact.

Solution. True

3. The function $f(x, y) = x\sin(y)$ is homogeneous.

Solution. False

4. The integrating factor of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 2x$$

is $\exp(x/2)$.

Solution. True

2 Long answer questions

This section contains two questions with three points each. Each question requires a detailed explanation. All inequalities (such as Gronwall's inequality) from the course can be assumed.

5. Construct a continuous solution to the problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y(x) = q(x)$$

where

$$p(x) = \begin{cases} \lambda_1 & \text{if } 0 \le x \le 1 \\ \lambda_2 & \text{if } x > 1 \end{cases} \quad \text{and} \quad q(x) = \begin{cases} \lambda_3 & \text{if } 0 \le x \le 1 \\ \lambda_4 & \text{if } x > 1. \end{cases}$$

Solution. The integrating factor for $0 \le x \le 1$ is $\mu = \exp(\lambda_1 x)$. An application of method of integrating factors lead to, for $0 \le x \le 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\exp(\lambda_1 x)y) = \lambda_3 \exp(\lambda_1 x).$$

This implies

$$y(x) = \frac{\lambda_3}{\lambda_1} + C_1 \exp(-\lambda_1 x).$$

Similarly a general solution on $1 < x$ is given by

$$y(x) = \frac{\lambda_4}{\lambda_2} + C_2 \exp(-\lambda_2 x).$$

Continuity at $x = 1$ show that

$$\frac{\lambda_4}{\lambda_2} + C_2 \exp(-\lambda_2) = \frac{\lambda_3}{\lambda_1} + C_1 \exp(-\lambda_1).$$

6. Let a, b be positive real numbers. Suppose that $f = f(x, y)$ is continuous on the rectangle

$$R = \{(x, y) : |x - x_0| \le a, |y - y_0| \le b\}$$

and is Lipschitz continuous with respect to y in R. Show that the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \quad \text{with} \quad y(x_0) = y_0$$

has at most one solution.

Solution. Follows from Gronwall's inequality.

3 Proof type question

This section contains only one question of five points. The answer should be stepwise, clear, and logical. Random attempts and gibberish solutions will be discarded.

7. A function f defined on a domain D of the real (t, x) plane is a Lipschitz(t) function if there exists an integrable function k of t such that for all (t, x) and $(t, \widetilde{x})$ in D

$$|f(t, x) - f(t, \widetilde{x})| \leq k(t)|x - \widetilde{x}|.$$

Let $f \in$ Lipschitz(t) on D. Let φ_1 and φ_2 be two continuous functions on the interval $I = [a, b]$ such that $(t, \varphi_j(t)) \in D$ for all $t \in I$, and $f(t, \varphi_j(t))$ is integrable over I for $j = 1, 2$. Let

$$\varphi_j(t) = \varphi_j(\tau) + \int_\tau^t f(s, \varphi_j(s))\mathrm{d}s + E_j(t)$$

where $t \in I$, and suppose that $|\varphi_1(\tau) - \varphi_2(\tau)| \leq \delta$. Prove, if $E(t) = |E_1(t)| + |E_2(t)|$, that for $\tau \leq t \leq b$,

$$|\varphi_1(t) - \varphi_2(t)| \leq \delta \exp\left(\int_\tau^t k(s)\,\mathrm{d}s\right) + E(t) + \int_\tau^t \left(E(s)k(s)\exp\left(\int_s^t k(u)\,\mathrm{d}u\right)\right)\mathrm{d}s.$$

Hint 1. Let φ, ψ, χ be real-valued continuous (or piecewise continuous) functions on a real t interval $I = [a, b]$. Let $\chi(t) > 0$ on I, and suppose for $t \in I$ that

$$\varphi(t) \leq \psi(t) + \int_a^t \chi(s)\varphi(s)\,\mathrm{d}s.$$

Then it holds

$$\varphi(t) \leq \psi(t) + \int_a^t \chi(s)\psi(s)\left(\int_s^t \chi(u)\mathrm{d}u\right)\mathrm{d}s.$$

Hint 2 (integrability) A function $g : I \to \mathbb{R}$ is integrable on I if the integral $\int_I g\,\mathrm{d}x$ is well defined and

$$\int_I |g|\,\mathrm{d}x < \infty.$$

Solution. Some elementary algebra shows that

$$|\varphi_1(t) - \varphi_2(t)| \leq \delta + \int_\tau^t k(s)|\varphi_1(s) - \varphi_2(s)|\,\mathrm{d}s + E(t).$$

An application of the Gronwall's inequality leads to

$$\begin{aligned}
|\varphi_1(t) - \varphi_2(t)| &\leq \delta + E(t) + \int_\tau^t k(s)(\delta + E(s))\exp\left(\int_s^t k(u)\,\mathrm{d}u\right)\mathrm{d}s \\
&= \delta\left(1 + \int_\tau^t k(s)\exp\left(\int_s^t k(u)\,\mathrm{d}u\right)\mathrm{d}s\right) + \int_\tau^t k(s)E(s)\exp\left(\int_s^t k(u)\,\mathrm{d}u\right)\mathrm{d}s.
\end{aligned}$$

Define a function

$$\begin{aligned}
y(t) &:= 1 + \int_\tau^t k(s)\exp\left(\int_s^t k(u)\,\mathrm{d}u\right)\mathrm{d}s \\
&= 1 + \exp\left(\int_\tau^t k(u)\,\mathrm{d}u\right)\int_\tau^t k(s)\exp\left(-\int_\tau^s k(u)\,\mathrm{d}u\right)\mathrm{d}s
\end{aligned}$$

Observe that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = &\ \exp\left(\int_\tau^t k(u)\,\mathrm{d}u\right) \times k(t) \exp\left(-\int_\tau^t k(u)\,\mathrm{d}u\right) + \\ &\int_\tau^t k(s) \exp\left(-\int_\tau^s k(u)\,\mathrm{d}u\right)\,\mathrm{d}s \times \exp\left(\int_\tau^t k(u)\,\mathrm{d}u\right) \times k(t) \\ &= k(t) + k(t) \times \int_\tau^t k(s) \exp\left(\int_s^t k(u)\,\mathrm{d}u\right)\,\mathrm{d}s \\ &= k(t) + k(t)(y(t) - 1) = k(t)y(t).\end{aligned}$$

Also we have $y(\tau) = 1$. This implies, $y(t) = \exp\left(\int_\tau^t k(s)\,\mathrm{d}s\right)$. Therefore,

$$|\varphi_1(t) - \varphi_2(t)| \leq\ = \delta \exp\left(\int_\tau^t k(s)\,\mathrm{d}s\right) + \int_\tau^t k(s)E(s) \exp\left(\int_s^t k(u)\,\mathrm{d}u\right)\,\mathrm{d}s.$$