Adjoint problems (cont^d)

$$h(x) = \frac{1}{a_0} \exp\left(\int^x \frac{a_1(t)}{a_0(t)}\, dt \right) \quad \text{if} \quad a_0(x) \neq 0 \quad \forall x$$

Recall $L(y) = a_0 y'' + a_1 y' + a_2 y$

Define a new operator $M(y) = h L(y) = h a_0 y'' + h a_1 y' + h a_2 y$

$$= b_0 y'' + b_1 y' + b_2 y$$

Note that $b_0' = \frac{d}{dx}(h a_0) = a_0 h' + h a_0'$

$$= a_0 \left(\frac{1}{a_0} \exp\left(\int^x \frac{a_1(t)}{a_0(t)} \right) \cdot \frac{a_1(x)}{a_0(x)} \cdots \right.$$

$$\left. + \exp\left(\int^x \frac{a_1(t)}{a_0(t)} \right)\left(\frac{-1}{a_0^2} \right) a_0'(x) \right) + \frac{1}{a_0} \exp\left(\int^x \frac{a_1(t)}{a_0(t)} \right) a_0'$$

$$= a_0 \left(\frac{a_1}{a_0^2} \exp\left(\int^x \frac{a_1(t)}{a_0(t)}\, dt \right) - \frac{a_0'(x)}{a_0^2} \exp\left(\int^x \frac{a_1(t)}{a_0(t)} \right) \right) + \frac{a_0'}{a_0} \exp\left(\int^x \frac{a_1(t)}{a_0(t)}\, dt \right)$$

$$= \frac{a_1}{a_0} \exp\left(\int^x \frac{a_1(t)}{a_0(t)}\, dt \right) \cancel{- \frac{a_0'(x)}{a_0} \exp\left(\int^x \frac{a_1(t)}{a_0(t)}\, dt \right)} \cancel{+ \frac{a_0'(x)}{a_0} \exp\left(\int^x \frac{a_1(t)}{a_0(t)} \right) dt}$$

$$= a_1 h = b_1$$

$$M(y) = b_0 y'' + b_1 y' + b_2 y = b_0 y'' + b_0' y' + b_2 y$$

$$= \frac{d}{dx}(b_0 y') + b_2 y,$$

which is in the self adjoint form.

Activity. Obtain the self-adjoint forms of the following equations.

- Legendre equation $\quad (1-x^2) y'' - 2x y' + n(n+1) y = 0$
- Bessel's equation $\quad x^2 y'' + x y' + (x^2 - \nu^2) y = 0.$

Lagrange's identity

$$zL(y) - yL^*(z) = \frac{d}{dx}\left(a_0(y'z - zy') + (a_1 - a_0)yz \right)$$

Green's identity

$$\int_a^b \left(zL(y) - yL^*(z) \right) dx = a_0(y'z - zy') + (a_1 - a_0)yz \Big|_{x=a}^{x=b}$$

Remark. If L is self-adjoint

$$\int_a^b \left(zL(y) - yL^*(z) \right) dx = a_0(y'z - zy') \Big|_{x=a}^{x=b}$$