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Adjoint Forms

Consider the generic second order linear operator

$$L[y] = a_0 y'' + a_1 y' + a_2 y \qquad (1)$$

for all $a < x < b$ and $y \in \mathcal{C}^2[a,b)$.

Let $z \in \mathcal{C}^2(a,b)$. Multiply (1) with z and integrate on (a,x)

$$\int_a^x z\, L[y(t)]\,dt = \int_a^x z(t)\, a_0(t) y''(t)\,dt + \int_a^x z(t) a_1(t) y'(t)\,dt + \int_a^x z(t) a_2(t) y(t)\,dt$$

$$= \int_a^x \left[(z(t) a_0(t) y')' - (z(t) a_0(t))' y'(t) \right] dt + \int_a^x \left[(z(t) a_1(t) y)' - (z(t) a_1(t))' y \right] dt + \int_a^x z(t) a_2(t) y(t)\,dt$$

$$= -\int_a^x (z(t) a_0(t))' y'(t)\,dt - \int_a^x (z(t) a_1(t))' y + \int_a^x z(t) a_2(t) y(t)\,dt + \Big[z(t) a_0(t) y'(t) + z(t) a_1(t) y(t) \Big]_a^x$$

Verify

$$= \int_a^x \left[(z(t) a_0(t))'' - (z(t) a_1(t))' + (z(t) a_2(t)) \right] y(t)\,dt + \Big[z(t) a_0(t) y'(t) - (z(t) a_0(t))' y(t) + z(t) a_1(t) y(t) \Big]_a^x$$

Define an operator

$$\boxed{\begin{aligned} L^*(z) &= (z a_0)'' - (z a_1)' + (z a_2) \\ &= (z' a_0 + a_0' z)' - z' a_1 - z a_1' + z a_2 \\ &= z'' a_0 + 2 a_0' z' + a_0'' z - z' a_1 - z a_1' + z a_2 \\ &= a_0 z'' + (2a_0' - a_1) z' + (a_0'' - a_1' + a_2) z \end{aligned}}$$

Adjoint (opr) of L

$$\boxed{L^*(z) = a_0 z'' + (2a_0' - a_1) z' + (a_0'' - a_1' + a_2) z}$$

Exc. Verify that $(L^*)^* = L$

Qn. Can we have $L^* = L$? In general no!

Necessary and sufficient condition for $L^* = L$

$$a_0 z'' + (2a_0' - a_1) z' + (a_0'' - a_1' + a_2) z = a_0'' z + a_1 z' + a_2 z$$

$\iff$ $2a_0' - a_1 = a_1$ & $a_0'' - a_1' + a_2 = a_2$

$\iff$ $a_0' = a_1$

In this case

$$L(y) = a_0 z'' + a_0' z' + a_2 z = (a_0 z')' + a_2 z = \frac{d}{dx}\left(a_0(x) \frac{dz}{dx} \right) + a_2 z$$

Remark. Consider

$$-\frac{d}{dx}\left(p(x) \frac{dy}{dx} \right) + q(x) y = 0 \qquad \text{(self adjoint problem)}$$

If $p(x) \neq 0$ $\forall x \in [a,b]$, regular Sturm–Liouville problem

If not, singular Sturm–Liouville problem.

Exc. Show that if $a_0(x) \neq 0$ $\forall x$, then the problem

$$L(y) = 0$$

can be transformed into self-adjoint form by multiplying with an appropriate f^n $h(x)$. ie, $\exists$ $h(x)$ st.

$$h(x) L(y) = 0$$

is self-adjoint. Find $h(x)$ and the corresponding self adjoint form.