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# An Elementary Proof that Reduced Row Echelon form of a Matrix is Unique

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Many results in a first course in linear algebra rely on the uniqueness of reduced echelon form of a given matrix. Most textbooks either omit the proof of this important result or use ideas and results in a proof that are not typically available when the result is introduced. For example, the proof [1] is relegated to an appendix and relies on the concept of linear dependence relations among columns, while the proof in [3] uses elementary matrices, permutation matrices, and block matrix calculations.

Since the first course in linear algebra is often a first introduction to higher level conceptual thinking in mathematics and the careful use of definitions, theorems, and proofs, it is desirable to have an elementary proof of the uniqueness of reduced echelon form that is accessible to students at the time the fact is presented using only the fact that the solution sets of linear systems represented by row equivalent augmented matrices are the same. Consequently, this proof can be offered as a supplement to the main narrative of the class and might spark some interest among potential future majors.

The ideas in this proof are not new (see [4] and [2], for example) but the idea of using augmented matrices in the argument is novel. The proof here is also written specifically for undergraduate students in their first linear algebra class.

**Theorem.** *Every matrix is row equivalent to a unique matrix in reduced row echelon form.*

*Proof.* Gauss-Jordan elimination tells us that every matrix is equivalent to at least one matrix in reduced row echelon form. If a matrix is equivalent to two such matrices, they must be row equivalent because elementary row operations are reversible. Consequently, it suffices to show that row equivalent matrices in reduced row echelon form must be equal. We use an iterative process on the columns to show this.

Suppose, then, that we have two matrices $A$ and $B$ in reduced row echelon form that are row equivalent. Write $\mathbf{a}_k$ for the $k$-th column of $A$ and $A_k$ for the submatrix of $A$ consisting of the first $k$ columns. Define $\mathbf{b}_k$ and $B_k$ similarly from $B$. Note that $A_k$ and $B_k$ are also in reduced row echelon form and that $A_k$ is row equivalent to $B_k$, as the same row operations that convert $A$ into $B$ also convert $A_k$ into $B_k$.

Start with the case $k = 1$. We have that $A_1$ and $B_1$ each consist of a single column and are in reduced row echelon form. The only single column matrices in reduced row echelon form are the zero column and a column with a 1 in the first entry with zeros below it. These are clearly not row equivalent since the zero column is only row equivalent to itself. Hence, $A_1 = B_1$.

Moving to $k = 2$, we have $A_2 = \begin{bmatrix} A_1 & \mathbf{a}_2 \end{bmatrix}$ and $B_2 = \begin{bmatrix} B_1 & \mathbf{b}_2 \end{bmatrix}$. We already know that $A_1 = B_1$ so it remains to show that $\mathbf{a}_2 = \mathbf{b}_2$. Consider $A_2$ and $B_2$ as augmented matrices of the linear systems $A_1\mathbf{x} = \mathbf{a}_2$ and $B_1\mathbf{x} = \mathbf{b}_2$. Since $A_2$ and $B_2$ are row equivalent, the two systems have the same solution set. If the solution set is empty, then the systems are both inconsistent and so the last column is a pivot column. It follows that the last nonzero row of $A_2$ and $B_2$ is of the form $\begin{bmatrix} 0 & 1 \end{bmatrix}$, so both $\mathbf{a}_2$ and $\mathbf{b}_2$ have a 1 in the same position with zeros elsewhere, hence, are equal. If, on the other hand, the solution set is nonempty, let $\mathbf{x}$ be a solution. Then $\mathbf{a}_2 = A_1\mathbf{x} = B_1\mathbf{x} = \mathbf{b}_2$ and so $A_2 = B_2$.

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Now that we know that $A_2 = B_2$, we can move to $k = 3$ and repeat the argument in the last paragraph, writing $A_3 = \begin{bmatrix} A_2 & \mathbf{a}_3 \end{bmatrix}$ and $B_3 = \begin{bmatrix} B_2 & \mathbf{b}_3 \end{bmatrix}$. As before $A_2\mathbf{x} = \mathbf{a}_3$ and $B_2\mathbf{x} = \mathbf{b}_3$ both have the same solution set since $A_3$ is row equivalent to $B_3$. If these systems are inconsistent, then the last nonzero row of $A_3$ and $B_3$ must be $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{a}_3$ and $\mathbf{b}_3$ again have a 1 in the same position with zeros elsewhere, hence are equal. If the solution set is consistent, then we use a solution $\mathbf{x}$ as above to conclude that $\mathbf{a}_3 = \mathbf{b}_3$. Continuing to iterate the argument for each subsequent column from left to right until we reach the last column then shows that $A = B$, as desired. ■

Note that this proof could be easily reframed by using mathematical induction on the number of columns in $A$ and $B$. However, students enrolled in a first linear algebra course may not have seen mathematical induction yet. The iterative proof given above, on the other hand, would be accessible to a student enrolled in such a class.

The iterative approach also illustrates how each pair of columns first appears as augmented columns of equivalent linear systems in order to show they are equal. They are then incorporated into the coefficient matrix in the next iteration. We illustrate the approach using $k = 2$, where

$$A_1 = B_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{or} \quad A_1 = B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

In the first case, we would have

$$A_2 = \left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \vdots & \\ 0 & 0 \end{array}\right] \quad \text{or} \quad A_2 = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ \vdots & \\ 0 & 0 \end{array}\right]$$

and in the second case we would have

$$A_2 = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ \vdots & \\ 0 & 0 \end{array}\right] \quad \text{or} \quad A_2 = \left[\begin{array}{c|c} 1 & * \\ 0 & 0 \\ \vdots & \\ 0 & 0 \end{array}\right].$$

with the same possibilities for $B_2$. The first choice in each case corresponds to an inconsistent system, forcing $A_2 = B_2$. The second choice in each case produces a consistent system with the value of $*$ in the second case being determined by the exact solution set of the corresponding linear system, again forcing $A_2 = B_2$.

## References


[1] Lay, D., Lay, S., McDonald, J. (2016). *Linear Algebra and its Applications*, 5th ed. Boston: Pearson.

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[4] Thomas, Y. (1984). The reduced row echelon form of a matrix is unique: A simple proof. *Math. Mag.* 57(2): 93–94.


 