Answer Key for Test 1

Question 1: Invertible Matrices (Rishit)

Let A and B be $n \times n$ matrices such that $AB = I_n$ (the identity matrix).

Part (a) (3 Marks) **Problem:** Show that if C is another $n \times n$ matrix such that $CA = I_n$, then $B = C$.

Solution:

Given:

1. $AB = I_n$
2. $CA = I_n$

To show: $B = C$.

Give 1 Mark if the student has written down what is given and what is to be proved. This is just to give grace marks. Do not cut any marks if they have not explicitly written this down.

Proof:

$$CAB = C(I_n) = C.$$

Give an additional Mark if the student has realized that we need to multiply both sides by C.

However, since $AB = I_n$, we also have:

$$CAB = I_n B = B.$$

Thus, $C = B$. This shows that any matrix C satisfying $CA = I_n$ must be equal to B.

Part (b) (2 Marks) **Problem:** Find an $n \times n$ invertible matrix P such that PA is in reduced row echelon form (RREF).

Solution:

To find an invertible matrix P such that PA is in RREF, perform Gaussian elimination on A. At each step, record the elementary row operations needed to transform A into its RREF. Each elementary operation can be represented by an elementary matrix E_i.

Let $P = E_k E_{k-1} \cdots E_1$, where each E_i represents an elementary row operation applied to A. The matrix P is the product of all these elementary matrices, and it is invertible because each E_i is invertible. Thus, PA will be in RREF.

Give 1 Mark if the student has stated that the matrix P is the product of the elementary matrices. Give full marks only if the have give a one line justification as to why P is invertible.

Part (c) (1 Mark) **Problem:** Show that the bottom row of P cannot be 0.

Solution:

If the bottom row of P is 0 then the bottom row of PP^{-1} is also zero which is a contradiction.

This part is trivial so you can be liberal. Give 1 Mark if anything vaguely resembling a justifiation is given.

Part (d) (3 Marks) **Problem:** Use the previous parts to conclude that $P = B$ and thus A is invertible.

Solution:

From part (b), P is an invertible matrix such that PA is in RREF. Since $AB = I_n$, we know that $PAB = P$ hence by the same logic as the last part, the bottom row of PA cannot be 0. Hence the RREF of A must be I_n. Therefore,

$$PA = I_n.$$

From part (a), if $CA = I_n$ for any C, then $C = B$. Hence, $P = B$.

Since P is invertible, and $B = P$, B is also invertible. Thus, A has both a left inverse (B) and a right inverse ($A^{-1} = B$), proving that A is invertible.

Give full marks if there is some reasonable justification as to why the RREF of A is I_n.

Question 2: Vector Spaces and Linear Independence (Vanmathi)

Part (a) (2 Marks) **Problem:** Let V be a vector space and let $v, w \in V$. Show that if v and w are linearly dependent, then either $v = cw$ or $w = cv$ for some $c \in \mathbb{R}$.

Solution:

If v and w are linearly dependent, there exist scalars $a, b \in \mathbb{R}$, not both zero, such that:

$$av + bw = 0.$$

If $a \neq 0$, we can solve for v:

$$v = -\frac{b}{a}w,$$

where $c = -\frac{b}{a}$.

If $b \neq 0$, we can solve for w:

$$w = -\frac{a}{b}v,$$

where $c = -\frac{a}{b}$.

Thus, either $v = cw$ or $w = cv$, depending on which scalar is nonzero.

I think there is really one way to solve this straightforward problem and it was probably done in class. Cut 1 mark if someone does a division by 0.

Part (b) (3 Marks) **Problem:** Show that we can find n vectors $v_1, \ldots, v_n \in \mathbb{R}^2$ such that any pair v_i, v_j, $i \neq j$, are linearly independent.

Solution:

To find n vectors $v_1, \ldots, v_n \in \mathbb{R}^2$ such that any pair v_i, v_j is linearly independent, consider the following:

Choose vectors that are not scalar multiples of each other.

Give 2 marks if student has realized that all we need to do is find n vectors that are not scalar multiples of each other. Give 2 mark if the student has stated this in terms of lying on the same line.

For example, select:

$$v_i = \begin{bmatrix} \cos\left(\frac{2\pi i}{n}\right) \\ \sin\left(\frac{2\pi i}{n}\right) \end{bmatrix}, \quad i = 1, 2, \ldots, n.$$

Each vector represents a unique point on the unit circle in $\mathbb{R}^2$. No two of these vectors are scalar multiples of each other, so any pair v_i, v_j with $i \neq j$ are linearly independent.

Give full marks if an explicit correct example is given. Be liberal and give full marks even if the justification is not adequate, provided the student has mentioned vectors that are not scalar multiples of each other.

Part (c) (3 Marks) **Problem:** Let $v_1, \ldots, v_k \in \mathbb{R}^m$ be such that $v_i \cdot v_j = 0$ (the dot product) whenever $i \neq j$. Show that $v_1, \ldots, v_k$ are linearly independent.

Solution:

Suppose $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ for some scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Taking the dot product of both sides with v_i, we get:

$$(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) \cdot v_i = 0 \cdot v_i = 0.$$

Give 2 marks if the student has done the above step.

Since $v_i \cdot v_j = 0$ for $i \neq j$, this simplifies to:

$$c_i (v_i \cdot v_i) = 0.$$

Since $v_i \cdot v_i > 0$ (as it is the dot product of a vector with itself, and $v_i \neq 0$), we must have $c_i = 0$.

I missed mentioning that $v_i \neq 0$. If someone has noticed this, give them full marks! Do not penalize if they have concluded $c_i = 0$ without realising that this requires $v_i \neq 0$.

Since this is true for all $i = 1, 2, \ldots, k$, all coefficients $c_i = 0$. Hence, $v_1, \ldots, v_k$ are linearly independent.

Question 3: Matrix Inverse using Row Operations (Abhishek)

Problem: Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{bmatrix}.$$

Solution:

To find the inverse of A, we perform row operations to transform A into the identity matrix I_3, simultaneously applying these operations to I_3.

Start with:

$$[A \,|\, I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Step-by-Step Row Operations:

1. **Row 1:** $R_1 \leftarrow \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

2. **Row 3:** $R_3 \leftarrow R_3 - 4R_1$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right].$$

3. **Row 2:** $R_2 \leftarrow \frac{1}{3}R_2$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right].$$

4. **Row 3:** $R_3 \leftarrow R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{10}{3} & -2 & \frac{1}{3} & 1 \end{array}\right].$$

5. **Row 3:** $R_3 \leftarrow -\frac{3}{10}R_3$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{10} & -\frac{3}{10} \end{array}\right].$$

6. **Row 2:** $R_2 \leftarrow R_2 + \frac{1}{3}R_3$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{10} & -\frac{3}{10} \end{array}\right].$$

7. **Row 1:** $R_1 \leftarrow R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & -\frac{1}{10} & \frac{1}{10} & \frac{3}{10} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{10} & -\frac{3}{10} \end{array}\right].$$

8. **Row 1:** $R_1 \leftarrow R_1 - \frac{1}{2}R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & 1 & \frac{3}{5} & -\frac{1}{10} & -\frac{3}{10} \end{array}\right].$$

The right side of the augmented matrix is now A^{-1}:

$$A^{-1} = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{10} \\ \frac{3}{5} & -\frac{1}{10} & -\frac{3}{10} \end{bmatrix}.$$

This is a simple computational problem. Give 1, 2 or 3 marks depending on how close they are to the correct answer.