Countability

08 October 2024 20:08

Ref. Methods of Real Analysis [Richard R. Goldberg]

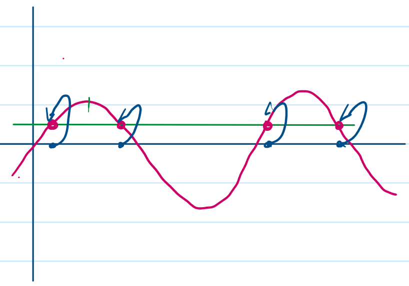
Self reading : Section 1.1 – 1.4

Recall. Definition (One-to-one function). If $f : A \to B$, then f is called a 1-1 function if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \quad \forall\, a_1, a_2 \in A$$

$\boxed{\text{Example:}}$ $f : \mathbb{R} \to \mathbb{R}$: $f(x) = \sin x$

$\sin(n\pi) = 0 \quad \forall\, n \in \mathbb{Z}$. Not a 1-1 fn.



Example: $\sin(\cdot) : [0, \pi/2] \to [0,1]$

$f(x) = x \quad f : \mathbb{R} \to \mathbb{R}$

Counter example. $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = (x^2, y^2)$$

Definition. If $f : A \to B$ is 1-1, then define a map $f^{-1} : \text{Range}(f) \to A$ such that

$$f^{-1}(y) = x \iff f(x) = y$$

Example. $f(x) = x^2$ $\quad \boxed{f : \mathbb{R}^+ \to \mathbb{R}^+}$

$$f^{-1}(y) = \sqrt{y} \iff f(\sqrt{y}) = (\sqrt{y})^2 = y$$

Note that

$$f[f^{-1}(y)] = y \quad \text{if } y \in \text{Range}(f)$$
$$f^{-1}[f(x)] = x \quad \text{if } x \in \text{Dom}(f)$$

Definition (Equivalence). If $f : A \to B$ is both 1-1 and onto, the f is called a bijection or we say that there exists a 1-1 correspondence between the elements of A and B. In this case, we say that A and B are equivalent sets.

Exc. Show that equivalence of sets define an equivalence relation on $2^{\mathbb{R}}$.

1. Let A and B are finite. and $A \equiv B$.

$$\text{Card}(B) \geq \text{Card}(A), \quad \text{Card}(A) \geq \text{Card}(B) \Rightarrow \text{Card}(B) = \text{Card}(A)$$

2. Let A is finite. Is it possible to have a 1-1 Correspondence between A and $B \subset A$?

Definition (infinite set). A set 'A' is said to be infinite, if there exists a bijection between A and a proper subset of it.

Examples. $\mathbb{N}$ $\quad f: \mathbb{N} \longrightarrow 2\mathbb{N}$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

$1 \longrightarrow 2$
$2 \longrightarrow 4$
$3 \longrightarrow 6$

Definition. A set A is said to be infinite, if for any given $n \in \mathbb{N}$, A contains a subset with precisely n elements.

Exc. Show that both definitions are equivalent.

Countable (denumerable) sets. A set A is said to be countable or denumerable if A is equivalent to $\mathbb{N}$, or A is finite. An uncountable set is an infinite set which is not countable.

Theorem. If $A_1, A_2, \ldots$ are countable sets, then $\bigcup_{i=1}^{\infty} A_i = A$ is also countable. (Countable union of countable sets is countable)

Proof. Since $A_1, A_2, \ldots$ are countable, we can enumerate their elements.

$$A_1 = \{a_1^1, a_2^1, a_3^1, \ldots\} \qquad A_i = \{a_1^i, a_2^i, \ldots\}$$

$$A_2 = \{a_1^2, a_2^2, a_3^2, \ldots\}$$

Diagonal argument

$$\begin{array}{ccccc}
a_1^1 & a_2^1 & a_3^1 & a_4^1 & \ldots \\
a_1^2 & a_2^2 & a_3^2 & a_4^2 & \ldots \\
a_1^3 & a_2^3 & a_3^3 & a_4^3 & \ldots \\
a_1^4 & a_2^4 & a_3^4 & a_4^4 & \ldots
\end{array}$$

$$a^4_1 \quad a^4_2 \quad a^4_3 \quad a^4_4 \quad \cdots$$

$f: A \longrightarrow \mathbb{N}$ $\qquad f(a^j_i) = 2^i 3^j$ (By prime factorization Thm $2^{i_1}3^{j_1} \neq 2^{i_2}3^{j_2}$ for $(i_1, j_1) \neq (i_2, j_2)$)

Theorem: $\mathbb{Q}$ is countable (Exc.)

Hint. $\mathbb{Q} = \bigcup_{i=1}^{\infty} A_i$, $\quad A_i = \left\{ \frac{i}{j} : j \in \mathbb{Z},\ j \neq 0 \right\}$

Theorem: If B is an infinite subset of a countable set A, the B is countable.

Real numbers

We assume that any real number x admits a decimal expansion. i.e.,

$$x = b.a_1 a_2 a_3 \cdots = b + a_1/10 + a_2/10^2 + a_3/10^3 + \cdots$$

where $b \in \mathbb{Z}$ and $a_i \in \{0, 1, \ldots, 9\}$

Theorem. The set $[0,1] = \{x : 0 \leq x \leq 1\}$ is uncountable.

Proof. Suppose $[0,1]$ is countable. Then $[0,1]$ admits an enumeration

$$x_1 = 0.a^1_1 a^1_2 a^1_3 \cdots$$

$$x_2 = 0.a^2_1 a^2_2 a^2_3 \cdots$$

$$x_3 = 0.a^3_1 a^3_2 a^3_3 \cdots$$

$$\vdots$$

$$x_n = 0.a^n_1 a^n_2 a^n_3 \cdots$$

Define $b = 0.b_1 b_2 b_3 \cdots$ $\qquad b_i = a^i_i + 1$ if $0 \leq a^i_i \leq 8$

$\qquad b_i = 0$ if $a^i_i = 9$

Construction $\Rightarrow$ $b \neq x_i \ \forall i$

Thus b is not in the enumeration. This shows that no enumeration of $[0,1]$ is possible.

As a consequence $\mathbb{R}$ is also uncountable ($[0,1] \subset \mathbb{R}$)

Boundedness

Definition. (Bounded above). A subset $A \subset \mathbb{R}$ is bounded above, if there exists a number $\mathbb{R} \ni M < \infty$ such that

$$x \leq M \quad \forall x \in A$$

Ex. $(-\infty, 1)$ $\quad \forall x \in (-\infty, 1) \quad x \leq 1$ Bounded above, not bdd below

$(1, \infty)$ Not bdd above, bdd below.

Bdd below. $\exists\ \mathbb{R} \ni m < \infty$ such that $x \geq m \quad \forall x \in A$

If A is both bdd above and below (ie, $\exists\ -\infty < m < M < \infty$ such that $A \subseteq [m, M]$) then A is called a bounded set.

- $|x| < M \quad \forall x \in A$
- $M > 0, \quad -M < x < M \quad \forall x \in A$

Definition. (Least upper bound or l.u.b or supremum). Let A is bounded above. A number L is called the least upper bound of A if

1. L is an upper bound of A.
2. If $\hat{L}$ is an upper bound, then $\hat{L} \geq L$

Defn. (Greatest lower bound or infimum) A is bdd below. Then u is called the infimum

1. u is a lower bound of A.
2. If $\hat{u}$ is a lower bound $\hat{u} \leq u$

> **Least upper bound axiom.** If A is a non empty, bdd above set, then A has a l.u.b. in $\mathbb{R}$.