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Diagonalizability of linear maps

Recall that $\lambda \in \mathbb{F}$ is said to be an eigenvalue of $T: V \to V$ if there exists a vector $0 \neq v$ such that

$$Tv = \lambda v$$

v is called an eigen vector corresponding to the eigenvalue λ.

Connection with matrices.

Let $A \in \mathbb{F}^{n \times n}$. Define $T_A: \mathbb{F}^n \to \mathbb{F}^n$ by

$$T_A(\vec{x}) = A \cdot \vec{x}$$

Suppose λ is an eigenvalue of A.

$$\Leftrightarrow \quad A\vec{v} = \lambda\vec{v} \quad \text{for some } 0 \neq v \in \mathbb{F}^n.$$

$$\Leftrightarrow \quad T_A \vec{v} = \lambda \vec{v}$$

$$\Leftrightarrow \quad \lambda \text{ is an eigenvalue of } T_A.$$

Suppose that V is a finite dimensional space of dimension 'n', and $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V.

Let $T: V \to V$ be a linear map.

Suppose that $[T]_B = A \in \mathrm{Mat}(n \times n)$ is the matrix representation of T.

$$T(v_j) = a_{1j} v_1 + \cdots + a_{nj} v_n$$

$$\begin{bmatrix} & a_{1j} & \\ & \vdots & 1 \leq j \leq n \\ & a_{nj} & \end{bmatrix}$$

This means, if $v = a_1 v_1 + \cdots + a_n v_n$

$[v]_B = (a_1, a_2, \dots, a_n)$. Then $\vec{y} = [T]_B [v]_B$ is the coordinate representation of Tv WRT B.

$$Tv = y_1 v_1 + \cdots + y_n v_n$$

Suppose that λ is an eigenvalue of $[T]_B = A$.

$$A\vec{x} = \lambda \vec{x} \quad \text{for some } 0 \neq \vec{x} \in \mathbb{F}^n.$$

$\Leftrightarrow$

$A\vec{z} = \lambda\vec{z}$ for some $0 \neq \vec{z} \in \mathbb{F}^n$.

$\Leftrightarrow$ let $\vec{z} = (z_1, z_2, \ldots, z_n)$ and $v_z = z_1 v_1 + \cdots + v_n z_n$ $\left([v_z]_B = \vec{z} \right)$

$\Leftrightarrow$ $[Tv_z]_B = [T]_B [v_z]_B = A\vec{z} = \lambda\vec{z}$

$\Leftrightarrow$ $Tv_z = \lambda z_1 v_1 + \cdots + \lambda z_n v_n = \lambda(z_1 v_1 + \cdots + z_n v_n)$
$= \lambda \vec{v}_z$

Summary. If $v \in V$ is an eigen vector of T, then $[v]_B$ is an eigenvector of $[T]_B$.

If $\vec{y} \in \mathbb{F}^n$ is an eigenvector of $[T]_B$, then $v_y = \sum_{i=1}^{n} v_i y_i$ $\left([v_y]_B = \vec{y} \right)$ is an eigenvector of T.

Example. $T: \mathbb{P}^3(\mathbb{R}) \to \mathbb{P}^3(\mathbb{R})$

$$T(a_3 x^3 + a_2 x^2 + a_1 x + a_0) = (a_3 + a_2)x^2 + (a_2 + a_1)x + a_0$$

$B = (x^3, x^2, x, 1)$

$$[T]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\lambda_1 = 1$ $(AM = 3)$ $\quad \lambda_2 = 0$ $(AM = 1)$

Eigenspace corresponding to $\lambda_1 = 1$

$$[T]_B - I = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R \leftrightarrow R$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot x_1, x_2

Free x_3, x_4

$\Rightarrow x_1 = 0, \; x_2 = 0$

$$E(\lambda_1) = \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

$$\hat{E}(\lambda_1) = \text{Span}\{x, 1\} \subseteq \mathbb{P}^3(\mathbb{R})$$

$$= \{c_1 x + c_2 : c_1, c_2 \in \mathbb{R}\}$$

$$T(c_1 x + c_2) = c_1 x + c_2$$

<u>Eigenspace of $\lambda_2 = 0$</u>

$$[T]_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \leftrightarrow R_1 \quad \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\equiv \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3 \quad \equiv \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\Rightarrow \quad x_4 = 0, \quad x_2 = -x_3, \quad x_1 = x_3$$

$$E(\lambda_1) = \left\{ \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} : x_3 \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\hat{E}(\lambda_1) = \operatorname{Span}\{ x^3 - x^2 + x \}$$

$$T(x^3 - x^2 + x) = 0 \cdot x^3 + 0 \cdot x + 0 = 0\,(x^3 - x^2 + x)$$

Definition. Let $T: V \to V$ be a linear map on a FDVS V. The map T is said to be diagonalizable if $\exists$ a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is diagonal.

$[T]_{B_1}$ is a diagonal.

$B_2 \to [T]_{B_2}$

$$[Tv]_{B_2} = [T]_{B_2}[v]_{B_2} = [T]_{B_2} C_{B_1}^{B_2} [v]_{B_1}$$

$$C_{B_1}^{B_2} [Tv]_{B_1} = C_{B_1}^{B_2} [T]_{B_1} [v]_{B_1}$$

$$C_{B_1}^{B_2} [T]_{B_1} [v]_{B_1} = [T]_{B_2} C_{B_1}^{B_2} [v]_{B_1}$$

$$[T]_{B_1} [v]_{B_1} = (C_{B_1}^{B_2})^{-1} [T]_{B_2} C_{B_1}^{B_2} [v]_{B_1}$$

$$[T]_{B_1} = (C_{B_1}^{B_2})^{-1} [T]_{B_2} C_{B_1}^{B_2}$$

$[T]_{B_1}$ and $[T]_{B_2}$ are similar