8th October

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Plan

- Spectral decomposition theorem
  [Complex eigen values and complex eigenvectors]
- Generalisation of dot product → Inner product
  " distance → norm
- Determinant, Rank → Existence of solutions for linear systems

1. Computation of complex eigenvalues and eigenvectors

Consider $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$

Characteristic polynomial $\chi_A(x) = \det(A - \lambda I)$

$$= \det \begin{pmatrix} 2-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}$$

$$= (2-\lambda)(1-\lambda) + 3$$

Solutions of $\chi_A(\lambda) = 0$ are

$$2 + \lambda^2 - 3\lambda + 3 = 0 \Rightarrow \lambda^2 - 3\lambda + 5 = 0$$

$$\lambda = \frac{3 \pm \sqrt{-11}}{2}$$

Two eigen values: $\lambda_1 = \frac{3+\sqrt{-11}}{2}$ $\lambda_2 = \frac{3-\sqrt{-11}}{2}$

Let us calculate the associated eigenvectors :

Case 1. $\lambda_1 = \frac{3+\sqrt{-11}}{2}$

Suppose that $v_1 = (x \; y)^T$ is an eigenvector corresponding to λ_1.

Then $Av_1 = \lambda_1 v_1$

$$\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{3+\sqrt{-11}}{2}\right) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \quad 2x - y - \left(\frac{3+\sqrt{-11}}{2}\right)x = 0 \longrightarrow (1)$$

$$3x + y - \left(\frac{3+\sqrt{-11}}{2}\right)y = 0 \longrightarrow (2)$$

$$2(2x - y) - 3x - \sqrt{-11}\,x = 0$$

$$\Rightarrow x - \sqrt{-11}\,x - 2y = 0$$

$$\Rightarrow y = \frac{x}{2}(1 - \sqrt{-11}) \longrightarrow (3)$$

$$2(3x + y) - (3 + \sqrt{-11})y = 0$$

$$-y + 6x - \sqrt{-11}\,y = 0$$

$$\Rightarrow x = \frac{y}{6}(1 + \sqrt{-11}) \longrightarrow (4)$$

(3) gives $v_1 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{-11}}{2} \end{pmatrix}$

(4) gives $\hat{v}_1 = \begin{pmatrix} \frac{1+\sqrt{-11}}{6} \\ 1 \end{pmatrix}$

Qn. What is the relation between v_1 and $\hat{v}_1$?

$$\left(\frac{1-\sqrt{-11}}{2}\right)\hat{v}_1 = \begin{pmatrix} \left(\frac{1+\sqrt{-11}}{6}\right)\left(\frac{1-\sqrt{-11}}{2}\right) \\ \frac{1-\sqrt{-11}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+11}{12} \\ \frac{1-\sqrt{-11}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \frac{1-\sqrt{-11}}{2} \end{pmatrix} = v_1$$

Eigenspace corresponding to λ_1, $E(\lambda_1) = \left\{ z \begin{bmatrix} 1 \\ \frac{1-\sqrt{-11}}{2} \end{bmatrix} : z \in \mathbb{C} \right\}$

Eigenvector corresponding to λ_2,

$$v_2 = \overline{v_1} = \begin{bmatrix} 1 \\ \end{bmatrix}$$

eigenvector corresponding to λ_2)

$$v_2 = \overline{v_1} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{-11}}{2} \end{bmatrix}$$

Eigenspace $E(\lambda_2) = \left\{ z \begin{bmatrix} 1 \\ \frac{1+\sqrt{-11}}{2} \end{bmatrix} : z \in \mathbb{C} \right\}$

Remark 1. If $\chi_A(\lambda)$ is a polynomial with real valued coefficients, the complex roots, if any, appear as complex conjugate pairs.

[If $\lambda \in \mathbb{C}$ is a root, then $\bar{\lambda} \in \mathbb{C}$ is also a root]

Pf. $\chi_A(\lambda) \in P^n(\mathbb{R})$ is given by

$$\chi_A(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$
$$(a_n \neq 0)$$

Since λ is a root,

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$
$$\overline{a_n\lambda^n} + \overline{a_{n-1}\lambda^{n-1}} + \cdots + \overline{a_1\lambda} + \overline{a_0} = 0$$
$$\overline{a_n}\,\overline{\lambda}^n + \overline{a_{n-1}}\,\overline{\lambda}^{n-1} + \cdots + \overline{a_1}\,\overline{\lambda} + \overline{a_0} = 0$$
$$a_n\overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = 0$$
$$\Rightarrow \chi_A(\overline{\lambda}) = 0$$

Easy way

$A\vec{x} = \lambda\vec{x}$; $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \Rightarrow$$

$2x - y - \lambda x = 0 \rightarrow ⑤$

$3x + y - \lambda y = 0 \rightarrow ⑥$

⑤ : $(2-\lambda)x - y = 0$

⑥ : $3x + (1-\lambda)y = 0$

$$v_1 = \begin{pmatrix} 1 \\ 2-\lambda_1 \end{pmatrix}$$

$$\hat{v}_1 = \begin{pmatrix} \frac{\lambda_1 - 1}{3} \\ 1 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 2-\lambda_2 \end{pmatrix} \qquad \hat{v}_2 = \begin{pmatrix} \frac{\lambda_2 - 1}{3} \\ 1 \end{pmatrix}$$

Quick check: $\lambda_1 = \frac{3+\sqrt{-11}}{2}$

$$v_1 = \begin{pmatrix} 1 \\ 2-\lambda_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 - \frac{3+\sqrt{-11}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{4-3-\sqrt{-11}}{2} \end{pmatrix}$$
(2)

$$= \begin{pmatrix} 1 \\ \frac{1-\sqrt{-11}}{2} \end{pmatrix}$$

Remark 2. Suppose that $v_1 = \begin{pmatrix} 1 \\ 2-\lambda_1 \end{pmatrix}$ is an eigen vector

$(x, y)^T$ should satisfy (7) with $\lambda = \lambda_1$ ($x = 1$, $y = 2-\lambda_1$)

$$(2-\lambda)x - y = 0 \qquad (7)$$

$$(2-\lambda_1) - (2-\lambda_1) = 0$$

$$3x + (1-\lambda_1)y = 0 \Rightarrow 3 + (1-\lambda_1)(2-\lambda_1) = 0$$

[$\because$ $3+(1-\lambda)(2-\lambda)$ is the CE and λ_1 is a root]

Computation using Gaussian elimination.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

characteristic polynomial $\chi_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$

$$= (1-\lambda)(1-\lambda)^2 - 1(0) + 1(1-\lambda)$$

$$= (1-\lambda)^3 + 1-\lambda = (1-\lambda)((1-\lambda)^2+1)$$

Roots of $\chi_A(\lambda) = 0$ are

$$(1-\lambda)\left((1-\lambda)^2+1\right) = 0 \Rightarrow \lambda_1 = 1$$

$$(1-\lambda)^2 = -1 \Rightarrow 1-\lambda = \pm i$$

$$\lambda_2 = 1+i$$

$$\lambda_3 = 1-i$$

Eigenvectors corresponding to $\lambda_1 = 1$

$$(A-\lambda_1 I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{0}$$

$\lambda_1 = 1$

$$A - I = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$R_1 \leftrightarrow R_3$

$$\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\equiv \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_1 = \begin{bmatrix} 0 \\ z \\ z \end{bmatrix} = z\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Eigenspace $E(\lambda_1) = \left\{ z\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{C} \right\}$

Eigen vectors corresponding to $\lambda_2 = 1+i$

$$A - (1+i)\mathbb{I}_3 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1+i \end{bmatrix}$$

$$= \begin{bmatrix} -i & 1 & -1 \\ 0 & -i & 0 \\ 1 & 0 & -i \end{bmatrix} \quad R_1 \to iR_1$$

$$\equiv \begin{bmatrix} 1 & i & -i \\ 0 & -i & 0 \\ 1 & 0 & -i \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\equiv \begin{bmatrix} 1 & i & -i \\ 0 & -i & 0 \\ 0 & -i & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\equiv \begin{bmatrix} 1 & i & -i \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 \times i$$

$$\equiv \begin{bmatrix} 1 & i & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$y = 0 \qquad x = i(z - y) = iz$

$$v_2 = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix} z \qquad v_3 = \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} \quad (\because v_3 = \overline{v_2})$$