Some properties of characteristic polynomial

Lemma. If A is a triangular matrix, then eigenvalues are the diagonal entries.

Pf. $\chi_A(\lambda) = (a_{11} - \lambda) \cdots (a_{nn} - \lambda)$ if A is triangular.

Therefore roots are $\lambda_1 = a_{11}, \ldots, \lambda = a_{nn}$.

Observation. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues.

Then $\chi_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

$$= \lambda^n - (\lambda_1 + \cdots + \lambda_n)\lambda^{n-1} + \cdots\cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$$

Example. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - \lambda \underbrace{\boxed{(a_{11} + a_{22})}}_{= \text{trace}(A)} + \underbrace{\boxed{a_{11}a_{22} - a_{12}a_{21}}}_{= \det(A)}$$

$$= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

Example. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix}$$

$$= (a_{11} - \lambda)\left[(a_{22} - \lambda)(a_{33} - \lambda) - a_{23}a_{32}\right] + \cdots$$

$$= (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) - a_{23}a_{32}(a_{11}-\lambda) + \cdots$$
$$= -\lambda^3 + (a_{11}+a_{22}+a_{33})\lambda^2 + \cdots + \cdots$$

In general the characteristic polynomial is of the form

$$\chi_A(\lambda) = (-1)^n(\lambda^n - (\operatorname{tr} A)\lambda^{n-1} + \cdots)$$
$$= (-1)^n(\lambda^n - (\lambda_1 + \cdots + \lambda_n)\lambda^{n-1} + \cdots)$$

Comparison gives $\boxed{\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n}$

Lemma: $\lambda_1\lambda_2\cdots\lambda_n = \det(A)$

Pf. $\chi_A(\lambda) = \det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$

Put $\lambda = 0 \Rightarrow \boxed{\det(A) = \lambda_1 \cdots \lambda_n}$

DIAGONALIZATION

Suppose that $A_{n\times n}$ has n linearly independent eigenvectors

$\Leftarrow$ n distinct eigenvalues

$\Leftrightarrow$ $\forall \lambda_i \quad GM(\lambda_i) = AM(\lambda_i)$

Let the eigenvectors are $v_1, \ldots, v_n$

$$Av_j = \lambda_j v_j$$

Define a matrix $P = [v_1 \; v_2 \cdots v_n]$

$$AP = A[v_1 \; v_2 \; \cdots \; v_n]$$
$$= [Av_1 \; Av_2 \; \cdots \; Av_n]$$
$$= [\lambda_1 v_1 \; \lambda_2 v_2 \; \cdots \; \lambda_n v_n]$$

Let $D = \text{diag}(\lambda_1 \cdots \lambda_n)$

$$AP = D[v_1 \; v_2 \; \cdots \; v_n]$$
$$= DP$$

Since $\text{rank}(P) = n$, P is invertible. Therefore

$$P^{-1}AP = D$$ (That is A is similar to $D = \text{diag}(\lambda_1 \cdots \lambda_n)$)

If A matrix has n linearly independent eigenvectors, then $\exists$ a matrix $P = [v_1 \cdots v_n]$, v_j are eigenvectors, and a diagonal matrix $D = \text{diag}(\lambda_1 \cdots \lambda_n)$ such that $P^{-1}AP = D$.

Def^n: If A is similar to a diagonal matrix, we say A is diagonalizable.

Thm. A is diagonalizable iff A has n linearly independent eigenvectors.

Pf. Suppose A is diagonalizable. Then $\exists$ P such that

$$P^{-1}AP = D$$
$$\Rightarrow AP = PD = DP$$
$$\Rightarrow AP = [d_1 p^1 \cdots d_n p^n]$$ p^j – Column of P
$$\Rightarrow [Ap^1 \cdots Ap^n] = [d_1 p^1 \cdots d_n p^n]$$
$$\Rightarrow Ap^j = d_j p^j$$
$$\Rightarrow p^j \text{ is an e. vector with eigenvalue } d_j$$

Since P is an inv. matrix, the columns are LI.