Complex eigenvalues (cont^{d}.) [10th october]

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Repeated eigenvalues

Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

Characteristic polynomial, $\chi_A(\lambda) = \det(A - \lambda I)$

$$= \det \begin{vmatrix} -\lambda & 1 & 1 \\ -1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (\lambda - 1)^2 (2 - \lambda) \quad \text{(Exc.)}$$

Eigenvalues are $\lambda_1 = 1$ $\quad (AM(\lambda_1) = 2; \quad GM(\lambda_1) = ?)$

$\lambda_2 = 2$ $\quad (AM(\lambda_2) = 1; \quad GM(\lambda_2) = ?)$

Eigenspace of $\lambda_1 = 1$

Consider the matrix $A - \lambda_1 I = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ (Recall $\lambda_1 = 1$)

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\equiv \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Free variables are y and z; $x = y + z$

Eigenspace, $E(\lambda_1) = \left\{ \begin{bmatrix} y+z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\}$

$$= \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim(E(\lambda_1)) = 2$ (There exists two linearly independent eigenvectors corresponding to λ_1)

$\therefore \quad GM(\lambda_1) = 2 = AM(\lambda_1)$

Exc. Find out the $GM(\lambda_2)$? Verify that

$$E(\lambda_2) = \left\{ z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}.$$

Summary: The matrix A has 2 LI e.vectors with eigenvalue $\lambda_1 = 1$ and 1 e.vector with eigenvalue $\lambda_2 = 2$.

These three vectors are linearly independent.

Repeated complex eigenvalues

Consider the matrix
$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Characteristic polynomial of A, $\chi_A(\lambda) = (\lambda^2+1)^2$

Eigenvalues: $\lambda_1 = i$ (2 times) $\quad \lambda_2 = -i$ (2 times)

Eigenspace $E(\lambda_1)$

$$A - \lambda_1 I = \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix} \xrightarrow[R_3 \to iR_3]{R_1 \to iR_1} \equiv \begin{bmatrix} 1 & -i & 0 & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & -i \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_3]{R_2 \to R_2 - R_1} \equiv \begin{bmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free variables are x_2 and x_4.

$$x_1 = i x_2 \qquad x_3 = i x_4$$

$$\text{Eigenspace} = \left\{ \begin{bmatrix} i x_2 \\ x_2 \\ i x_4 \\ x_4 \end{bmatrix} : x_2, x_4 \in \mathbb{C} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} \right\}$$

$$GM(\lambda_1) = 2$$

$$E(\lambda_2) = \text{Span} \left\{ \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -i \\ 1 \end{bmatrix} \right\}$$

Remarks.

Thm. (Fundamental Theorem of algebra) Every nonconstant polynomial $P_n(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0$, where $c_j \in \mathbb{C}$, has at least one root in $\mathbb{C}$.

As a consequence every matrix has at least one eigenvalue in $\mathbb{C}$.