Linear algebra and series

Assignment 4

(Matrix representation (contd) and eigenvalues)

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1 Tutorial questions

1. Let V be a two dimensional vector space over the field F, and B be an ordered basis for V. If T is a linear operator on V and

$$[T]_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then prove that $T^2 - (a + d)T + (ad - bc)I = 0$.

2. Let T be a linear operator on $\mathbb{R}^3$, the matrix of which in the standard basis is

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}.$$

Find a basis for the range of T and a basis for the nullspace of T.

3. Find the eigenvalues of the rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. For what values of θ are the eigenvalues real?

4. Let $A \in \mathbb{F}^{n\times n}$ and λ be an eigenvalue of A. Suppose $f(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is a polynomial with coefficients from $\mathbb{F}$. Show that if $Av = \lambda v$ for some $v \in \mathbb{F}^n$ then

$$(a_n A^n + a_{n-1}A^{n-1} + \cdots + a_0 I)v = f(\lambda)v.$$

5. If each row sum of a matrix A is 1, show that 1 is an eigenvalue of A.

6. For a linear operator $T : V \to V$, a linear subspace U of V is said to be an invariant subspace (or explicitly T-invariant), if $T(U) \subset U$. Must every linear operator have a proper invariant subspace? Next, suppose T has an eigenvalue λ; verify that the eigenspace E_λ comprising eigenvectors associated to the eigenvalue λ is T-invariant.

7. Let V be vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$ which are continuous. Let T be the linear operator on V defined by

$$(Tf)(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Prove that T has no eigenvalues.

2 Exercises

1. Given that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with $T((2,3)) = (2,2,1)$ and $T((3,4)) = (-1,0,2)$, write down the matrix representation of T.

2. Find the eigenvalue(s) and eigenvectors, if any, of the operator $T(x,y) = (2x + 4y, 2y)$ on $\mathbb{R}^2$.

3. Find the eigenvalues and bases for the corresponding eigenspaces of the following matrices.

   (a) $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

   (c) $\begin{bmatrix} -4 & -4 & 2 \\ 3 & 4 & -1 \\ -3 & -2 & 3 \end{bmatrix}$

4. Find the eigenvalues and eigenvectors of the differential operator $\dfrac{\mathrm{d}}{\mathrm{d}x}$ on the polynomial space $P^k([0,1])$.

5. Prove that A is a singular matrix if and only if 0 is an eigenvalue.

6. Prove that the eigenvalues of an upper triangular (or lower triangular) matrix are its diagonal entries.

7. Define linear transformations A and B on $\mathbb{R}^2$ by

   $$A(x,y) = (x+y, y) \quad \text{and} \quad B(x,y) = (x+y, x-y)$$

   Find all eigenvalues of A and B and their eigenspaces.

8. Let A be a $n \times n$ matrix such that $A^k = 0$ for some $k \geq 1$. What are all the eigenvalues of A?

9. Let $T : V \to V$ be a linear transformation on a vector space V. If v is an eigenvector of T with eigenvalue λ, show that $T(v)$ is also an eigenvector of T with the same eigenvalue.