Linear algebra and series

Assignment 3

(Rank, Linear transformations, Kernel, Image)

August–November Semester 2024

Department of Mathematics, Indian Institute of Technology Palakkad

18 September, 2024 (Wednesday)

Instructors. Dr Gopikrishnan Chirappurathu Remesan, Dr Balakumar G P, and Dr Jaikrishnan Janardhanan

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1 Tutorial questions

1. Show that the solution set to the linear system $A\boldsymbol{x} = 0$ is a vector space of dimension $n - \mathrm{rank}(A)$ for any $m \times n$ matrix A over $\mathbb{R}$ or $\mathbb{C}$.

2. Let A and B be $n \times n$ matrices. Show that for any $n \times n$ matrix X
$$\mathrm{rank}\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} \geq \mathrm{rank}(A) + \mathrm{rank}(B).$$
Discuss the cases where $X = 0$ and $X = I$ respectively.

3. Find the dimensions of $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$, and find their bases for the linear transformation A on $\mathbb{R}^3$ by
$$A(x, y, z) = (x - 2z, y + z, 0)^{\mathrm{T}}.$$

4. **(Integral transformation).** Define the indefinite integral map $\mathrm{T}_{\mathrm{int}} : \mathrm{P}^k(\mathbb{R}) \to \mathrm{P}^{k+1}(\mathbb{R})$ by
$$\mathrm{T}_{\mathrm{int}}(f) = \int f(x)\,\mathrm{d}x \quad \text{for all} \quad f \in \mathrm{P}^k(\mathbb{R}).$$
Compute the kernel and range of $\mathrm{T}_{\mathrm{diff}}$ and verify the rank–nullity theorem. Compute the matrix representation of $\mathrm{T}_{\mathrm{int}}$ with respect to the natural bases of $\mathrm{P}^k(\mathbb{R})$ and $\mathrm{P}^{k+1}(\mathbb{R})$.

5. (**Rotation matrices**.) Suppose that $A \in \mathrm{R}^{2\times 2}$ rotates the vector $\boldsymbol{v} = (a, b)^{\mathrm{T}}$ by θ radians.
   (a) Compute the entries of A.
   (b) Write the linear transformation $\mathrm{T}_{\mathrm{rot}} : \mathbb{R}^2 \to \mathbb{R}^2$ that corresponds to A.
   (c) Let $\mathrm{B}_2 = \{(f_{11}, f_{12}), (f_{21}, f_{22})\}$ be an arbitrary basis of $\mathbb{R}^2$. Compute the change of basis matrices from $\mathrm{B}_1 = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ to B_2 and vice versa.
   (d) Compute the matrix representation of $\mathrm{T}_{\mathrm{rot}}$ with respect to B_2.
   (e) Compute the matrix representaion of $\mathrm{T}_{\mathrm{rot}}$ when viewed as a map from $\mathbb{R}^2$ with respect to B_2 to $\mathbb{R}^2$ with respect to B_1.

6. **(Reflection matrix).** Write the linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$ that reflects a vector about an axis tilted ϑ radians in the anticlockwise direction. Compute the matrix representation of this linear map.

7. Let $\mathrm{T}_{\mathrm{trans}} : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathrm{Mat}_{2\times 2}(\mathbb{R}$ by $\mathrm{T}_{\mathrm{trans}}(A) = \mathrm{A}^{\mathrm{T}}$. Show that there does not exists any matrix M such that $MA = \mathrm{T}_{\mathrm{trans}}(A)$.

2 Exercises

1. Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 0 \end{pmatrix}$$

   using elementary row operations.

2. Find the rank and a basis of the row space of each of the following matrices.

   (a)

$$\begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   (b)

$$\begin{bmatrix} 5 & -2 & 1 & 0 \\ -2 & 0 & -4 & 1 \\ 1 & -4 & -11 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

3. Show that rank of $B^t A^t$ = rank of AB, assuming that the product AB is defined.

4. Show that if A is not a square matrix, then rows of A form a linearly dependent set or the columns of A form a linearly dependent set.

5. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, x_2 + 2x_1, -x_1 - 2x_2 + 2x_3).$$

   (a) Verify that it defines a linear operator.

   (b) Compute the kernel (null-space) of T.

   (c) Compute the rank and nullity of T.

6. Let $T : V \longrightarrow V$ be a linear map. Prove that if $T^2 = T \circ T$ is injective, then T is injective.

7. If S, T are linear operators from V into V, show that the composition $S \circ T$ is also a linear operator from V into V.

8. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. It is known that $T((2,1)) = (3,4)$ and $T(1,1) = (-1,2)$. Determine $T((4,3))$.

9. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear transformation given by

$$T(x, y, z) = (2x - y + 5z, -4x + 2y - 10z).$$

   Write down a basis for the kernel (null-space) of T. Is T injective?

10. Let $V = \mathcal{P}^n([0,1])$. Compute the rank and nullity of the linear operators T_k given by $T_k(p) = p^{(k)}$ for every polynomial in $\mathcal{P}^n([0,1])$.Here $p^{(k)}$ denotes the $k^t h$ derivative of p.

11. Compute the rank and nullity of the trace operator from $M_n(\mathbb{R})$ into $\mathbb{R}$.

12. Does there exists a surjective linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$?

13. Can a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ map a line to a circle?

14. Is the function from $P_2(\mathbb{R})$ to $P_4(\mathbb{R})$ that squares a polynomial a linear transformation? What about the function that multiplies a given polynomial by x^2?

15. Let V, W be finite-dimensonal vector spaces and let U be a subspace of V. Show that

    (a) Show that any linear map $T : U \to W$ can be extended to a linear map $\widetilde{T} : V \to W$.

    (b) Assume that $\mathsf{dim}(W) \geq \mathsf{dim}(V)$. Show that we can find a linear map $T : V \to W$ whose kernel is precisely U.

16. by $T(A) = A \cdot C - C \cdot A$. Verify that it is a linear operator. What is the kernel of $tr \circ T$ where tr is the trace operator?

17. Let V be a (finite dimensional) vector space and W be a subspace of V. We say that an operator P on V is idempotent if $P^2 = P \circ P = P$. Show that there exists an idempotent operator P on V such that the range of P is W [range of P is the same as Image of P].

18. Let $P : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $P(A) = \frac{A+A^t}{2}$. Then show that P is a linear operator. Compute the rank and nullity of P.

19. Using the Rank-Nullity theorem prove that

$$Column \ \ rank \ \ of \ \ A = Row \ \ rank \ \ of \ \ A$$

    for an $m \times n$ matrix A with entries from $\mathbb{F}$.

20. Let V and W be two vector spaces over $\mathbb{F}$. Let

$$L(V, W) = \{T : V \to W : T \quad \text{is} \quad \text{a} \quad \text{linear} \quad \text{transformation}\}.$$

    Show that $L(V, W)$ is a vector space over $\mathbb{F}$ with the following operations,

$$(S + T)(v) = S(v) + T(v) \ \ and$$

$$(cT)(v) = cT(v)$$

    for any $S, T \in L(V, W)$, $c \in \mathbb{F}$ and $v \in V$.