Linear algebra and series

Assignment 2

(Vector spaces, subspaces, span, linear dependency, basis, LU decomposition, fundamental spaces)

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1 Tutorial questions

1. Let V be a vector space over $\mathbb{R}$. If $x_1, x_2, \cdots, x_n \in V$ be such that $\sum_j a_j x_j = 0$ some scalars $a_1, \ldots, a_n$ with $a_1 a_n \neq 0$, show that $\operatorname{span}(x_1, \ldots, x_{n-1}) = \operatorname{span}(x_2, \ldots, x_n)$.

2. Find three vectors in $\mathbb{R}^n$ which are linearly dependent, and are such that any two of them are linearly independent.

3. Let $W = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0\}$. Exhibit a basis and compute its dimension.

4. Let V_1 and V_2 be subspaces of a vector space of finite dimension such that $\dim(V_1 + V_2) = \dim(V_1 \cap V_2) + 1$. Show that $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

5. Show that an invertible matrix need not have an LU factorization/decomposition, by demonstrating this (i.e., proving the impossibility) for the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

6. Let A, B be the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Verify the following LU-factorizations

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix}.$$

Now, use these factorizations to show that $Bx = y$ has a solution for every $y \in \mathbb{C}^2$ whereas $Ax = y \in \mathbb{C}^3$ has a solution if and only if $y_1 - 2y_2 + y_3 = 0$; verify that $(1, -2, 1)$ lies in the null-space of A.

(Remark: LU and PLU decompositions are useful when we are required to solve multiple systems $Ax = b$ where the 'coefficient' matrix A remains the same but the RHS vector b keeps changing).

7. Let V be the set of all real numbers of the form $a + b\sqrt{2} + c\sqrt{3}$, where a, b, c are rational numbers. Show that V is a vector space over the rational number field $\mathbb{Q}$. Exhibit a basis for V.

8. Find the dimension and a basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

9. Let A and B be two complex matrices. Recall that $R(A)$ is the range (column) space of A and $R(A^{\mathrm{T}})$ is the row space of A. Show that $R(A) \subseteq R(B) \Leftrightarrow A = BC$ for some matrix C and $R(A^{\mathrm{T}}) \subseteq R(B^{\mathrm{T}}) \Leftrightarrow A = RB$ for some matrix B.

10. Let A and B be $m \times n$ matrices. Show that $R(A + B) \subseteq R(A) + R(B)$.

2 Exercises

1. Verify that the set of all symmetric $n \times n$ matrices, i.e. the set of matrices $A = (a_{ij})_{n\times n}$ such that $a_{ij} = a_{ji}$ for all $i, j = 1, 2, \cdots, n$ is a linear space.

2. Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis of V which has four elements.

3. Let V be the vector space of all 2×2 matrices over the field F. Let W_1 be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}.$$

   (a) Prove that W_1 and W_2 are subspaces of V.
   (b) Find the dimensions of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$.

4. Determine the dimension of the vector space spanned by

$$\{(1, -3, 8, -3), (-2, 4, 6, 0), (0, 1, 5, 7)\}.$$

5. Let A be a matrix of order 3×3 with real entries. Suppose A commutes with every matrix B of order 3×3 with real entries. (This means $AB = BA$.) Show that A must be a scalar matrix, that is, a scalar multiple of the identity matrix.

6. Find 3 different bases for $\mathrm{Mat}_{n\times n}(\mathbb{R})$, the space of $n \times n$ matrices over the field $\mathbb{R}$.

7. Find a basis $\{A, B, C, D\}$ of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ such that $A^2 = A, B^2 = B, C^2 = C, D^2 = D$.

8. Find the span of

   (a) $1 + x^2$, $x + x^2$ and $1 + x + x^2$ in $P^2(\mathbb{R})$.
   (b) $1 - x^2$, $x - x^2$ and $2 - x - x^2$ in $W = \{p \in P^2(\mathbb{R}) :$ sum of the coefficients is $0\}$.

9. Determine whether each of the following form a linearly independent set of vectors:
    (a) $\{(1,2,6),(-1,3,4),(-1,-4,2)\}$ in $\mathbb{R}^3$
    (b) $\{(1,0,2,1),(1,3,2,1),(4,1,2,2)\}$ in $\mathbb{R}^4$
    (c) $\{u+v, v+w, w+u\}$ given that $\{u,v,w\}$ is a linearly independent set of vectors in $\mathbb{R}^{2023}$
10. In each of the following, determine whether the given set forms a basis:
    (a) $\{(5,3,7),(1,-3,6),(0,3,1)\}$ in $\mathbb{R}^3$.
    (b) $\{(1,0,0,0),(1,1,0,0),(1,0,1,0),(1,0,0,1)\}$ in $\mathbb{R}^4$.
    (c) $\{(1,0,-1),(1,2,1),(0,-3,2)\}$ in $\mathbb{R}^3$.
11. Find a basis and then the dimension for the following:
    (a) Subspace V_1 of $\mathbb{R}^3$ described by the equation $2x+3y-4z=0$,
    (b) Subspace of $\mathbb{R}^4$ given by $V_2 = \{(a,b,c,d,e) : a=c=e, b+d=0\}$
12. Show that the following vectors form a base for the space $\mathbb{R}^n$. $(1,1,1,1\cdots,1)$, $(0,1,0,0,\cdots,0)$; $(0,1,1,0,\cdots,0)$, $(0,1,1,1,0,\cdots,0)$, $\cdots$, $(0,1,1,1,\cdots,1)$
13. Find 3 different bases for $M_n(\mathbb{R})$.
14. Compute the dimension of the following subspaces of $\mathcal{P}^n([0,1])$.
    (a) $\{p \in \mathcal{P}^n([0,1]) : p(0)=0\}$
    (b) $\{p \in \mathcal{P}^n([0,1]) : p(0)=0 \text{ and } p(1)=0\}$
15. Exhibit a basis for the subspace consisting of all (a) upper triangular matrices, (b) diagonal matrices, (c) symmetric matrices in $M_n(\mathbb{R})$.
16. If $\{u_1, u_2, \cdots, u_n\}$ is a basis for $\mathbb{R}^n$, does it follow that $\{u_1-u_2, u_2-u_3, \cdots, u_{n-1}-u_n, u_n-u_1\}$ forms a basis of $\mathbb{R}^n$?
17. If V_1, V_2 are two vector spaces then show that $W = \{(v_1,v_2) : v_1 \in V_1 \text{ and } v_2 \in V_2\}$ is a vector space. Show that $dim(W) = dim(V_1) + dim(V_2)$.
18. Find a basis of the vector space $\mathbb{C}$ over $\mathbb{R}$.
19. Find a basis $\{A,B,C,D\}$ of $M_2(\mathbb{R})$ such that $A^2=A, B^2=B, C^2=C, D^2=D$.
20. Why is there no matrix whose row space and nullspace both contain $(1,1,1)$?
21. If the matrix A has the same four fundamental subspaces as B, does $A=cB$?