Linear algebra and series

Assignment 1

(Matrix theory, REF, RREF, Gaussian Elimination)

August–November Semester 2024

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19 August, 2024 (Monday)

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1 Tutorial

1. Let $A_{m\times n}$ and $B_{n\times \ell}$ be two matrices over a field $\mathbb{F}$. The rows of A are denoted by A_i and the columns of B by B^j with $1 \le i \le m$ and $1 \le j \le \ell$. Show that the product AB can be computed as

$$AB = \begin{pmatrix} A_1B \\ A_2B \\ \vdots \\ A_mB \end{pmatrix} = \begin{pmatrix} AB_1 & AB_2 & \cdots & AB_\ell \end{pmatrix}.$$

2. Suppose that $\mathcal{E}$ denotes an elementary row transformation (swapping $A_i \leftrightarrow A_j$, scaling $A_i \to A_j$, linear addition $A_j \to A_j + cA_i$). A matrix $E_{m\times m}$ obtained from the identity matrix $I_{m\times m}$ by applying a single elementary transformation is called an *elementary matrix*. Show that for a general matrix $A_{m\times m}$, it holds

$$\mathcal{E}(A) = EA.$$

3. Exercises 33–37 in **Chapter 2, Exercises II** from Introduction to Linear Algebra, S. Lang.

4. Consider the equation

$$x_1\vec{\boldsymbol{A}}_1 + \cdots + x_n\vec{\boldsymbol{A}}_n = \vec{0}, \tag{1}$$

where $\vec{\boldsymbol{A}}_1, \ldots, \vec{\boldsymbol{A}}_n$ are column vectors of size $m \times 1$.

   (a) Show that if all rows of the RREF of the matrix $\begin{bmatrix} \vec{\boldsymbol{A}}_1 & \ldots & \vec{\boldsymbol{A}}_n \end{bmatrix}$ are nonzero, then $x_1 = x_2 = \cdots = x_n = 0$. Otherwise, show that there exists infinitely many solutions that satisfy (1).

   (b) Suppose that

$$\vec{\boldsymbol{A}}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{\boldsymbol{A}}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\boldsymbol{A}}_3 = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}.$$

   Show that $\vec{\boldsymbol{A}}_3$ lives in the plane spanned by $\vec{\boldsymbol{A}}_1$ and $\vec{\boldsymbol{A}}_2$ by solving an appropriate system of linear equations.

5. Find the value of a for which Gaussian elimination leads to an inconsistent RREF for the system

$$\begin{aligned} au + v &= 1 \\ 4u + av &= 2. \end{aligned}$$

6. Solve by the Gaussian/Gauss-Jordan elimination algorithm the system of linear equation given by:

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 4, \\ 2x_1 - 3x_2 + 4x_3 - 3x_4 &= -1, \\ 3x_1 - 5x_2 + 5x_3 - 4x_4 &= 3, \\ -x_1 + x_2 - 3x_3 + 2x_4 &= 5. \end{aligned}$$

   After row-reducing the augmented matrix of the above system to its RREF, identify the pivot and free variables. Express the solution as a sum of a particular solution to the above system with solutions of the associated homogeneous system; express the solutions of the latter as a linear combination of a pair of linearly independent vectors in $\mathbb{R}^4$.

7. Suppose that a square matrix A can be row-reduced to a matrix in REF, without any row exchanges. Show that A can be factorized as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix.

2 Exercises

1. Let $A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & -1 \end{pmatrix}$. Compute ABC and CAB.

2. Suppose that $A = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}$. Verify that $(A^{\mathrm{T}}A)B = A^{\mathrm{T}}(AB)$.

3. Let A be a matrix of order 3×3 with real entries. Suppose A commutes with every matrix B of order 3×3 with real entries ($AB = BA$.) Show that A must be a scalar matrix, that is, a scalar multiple of the identity matrix.

4. The following statements involve matrices A, B and C that are assumed to be of appropriate size. Determine whether each of the statements is **TRUE** of **FALSE**.

   (a) If $A = B$, then $AC = BC$.

   (b) If $AC = BC$, then $A = B$.

   (c) If $AB = 0$, then $A = 0$ or $B = 0$.

   (d) If $A + C = B + C$, then $A = B$.

   (e) If $A^2 = I$, then $A = \pm I$.

   (f) If $AB = C$ and if two of the matrices are square, then so is the third.

   (g) If $AB = C$ and if C is a column vector, then so is B.

   (h) If $A^2 = I$, then $A^n = I$ for all integers $n \geq 2$.

   (i) If $A^2 = I$, then $A^n = I$ for all even integers $n \geq 2$.

   (j) If A and B are square matrices, then

$$(A + B)^2 = A^2 + B^2 + 2AB.$$

   (k) If A and B are square matrices, then

$$(A + B)(A - B) = A^2 - B^2.$$

5. Use Gauss Elimination Method to solve the following linear systems.

(a)

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 8 \\ 2x_1 + x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 + 4x_3 &= 11 \end{aligned}$$

(b)

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -5 \\ x_1 + x_2 - 3x_3 &= -9 \\ x_1 + 4x_2 + 2x_3 &= 9 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 + x_4 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ 4x_1 - x_2 - 2x_3 + 2x_4 &= 0 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \end{aligned}$$

(d)

$$\begin{aligned} x_1 + x_2 + x_4 &= 2 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \end{aligned}$$

6. Given the linear system

$$\begin{aligned} 2x_1 - 6\alpha x_2 &= 3 \\ 6\alpha x_1 - 2x_2 &= 3 \end{aligned}$$

(a) Find value(s) of α for which the system has no solution.

(b) Find value(s) of α for which the system has infinitely many solutions.

(c) Assuming a unique solution exists for a given α, find the solution.

7. Given the linear system

$$\begin{aligned} 2x_1 - x_2 + \alpha x_3 &= -2 \\ -x_1 + 2x_2 - \alpha x_3 &= 3 \\ \alpha x_1 + x_2 + x_3 &= 2 \end{aligned}$$

(a) Find value(s) of α for which the system has no solution.

(b) Find value(s) of α for which the system has infinitely many solutions.

(c) Assuming a unique solution exists for a given α, find the solution.