AN ANALYTICAL AND NUMERICAL STUDY OF A CLASS OF INTEGRO DIFFERENTIAL EQUATIONS WIT DELAY

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by

Arathi

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to

School of Mathematics

Indian Institute of Science Education and Research

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April 2015

CERTIFICATE

This is to certify that the work contained in this project report entitled **"A theoretical and numerical study of stochastic delay integro differential equations"** submitted by **Gopikrishnan C. R.** (**Roll No: IMS10026**) to Indian Institute of Science Education and Research Thiruvananthapuram towards partial requirement of **Master of Science** in Mathematics has been carried out by him under my supervision and that it has not been submitted elsewhere for the award of any degree.

Thiruvananthapuram - 695 016 | **Dr. M. P. Rajan**
April 2015 | **Project Advisor**

DECLARATION

I, **Gopikrishnan C. R.**, hereby declare that, this report entitled **"A theoretical and numerical study of stochastic integro differential equations"** submitted to Indian Institute of Science Education and Research Thiruvananthapuram towards partial requirement of **Master of Science** in Mathematics is an original work carried out by me under the supervision of **Dr. M. P. Rajan** and has not formed the basis for the award of any other degree or diploma, in this or any other institution or university. I have sincerely tried to uphold the academic ethics and honesty. Whenever an external information or statement or result is used they have been duly acknowledged and cited.

Thiruvananthapuram - 695 016 **Gopikrishnan C. R.**

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ABSTRACT

Among numerous volatility estimation schemes from historically available data, a class of discretization methods namely, *Generalized Auto Regressive Conditional Heteroskedasticity process* (GARCH) are of extreme importance because of the simplicity as well as robustness they posses. Therefore, it is important to understand how this discretized scheme behave in a continuously extended fashion. A recent upsurge in the publications from this area of quantitative finance corroborates the above a argument. Though several continuous time extensions have been proposed for GARCH process under varying assumptions and parameterizations, one which seems to be more natural is the delay integro differential formalism. Though deterministic and stochastic integro differential equations are well studied, little has been surveyed on stochastic delay integro differential equations (SDIDE). The main objective of this thesis is to establish important properties of stochastic delay integro differential equations. We have attempted to prove an analogous existence and uniqueness theorem of solutions for stochastic delay integro differential equations. Other important properties of the solutions like $\mathbb{L}^p$ boundedness and stability is also studied. Since it is nearly impossible to solve such equations analytically we tried to give approximate numerical solutions of a class of equations using the well celebrated Euler - Maruyama scheme and established an error bound for the solutions. Using the above tools, we tried to solve a particular SDIDE in mathematical finance which models the volatility, where we conclude this project.

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General Notation

positive : > 0.

non positive : ≤ 0.

negative : < 0.

non negative : ≥ 0.

$a.s$: almost surely.

$a.e$: almost everywhere.

$\mathbb{I}_A$: Indicator function (or characteristic function) of a set A.

$\mathbb{R}$: Set of all real numbers.

$\mathbb{R}^d$: $\{(x_1, x_2, \ldots, x_d) | x_i \in \mathbb{R} \quad \forall\, 1 \leq i \leq d\}$, d- dimensional real space.

$\mathbb{R}^+$: set of all non negative real numbers.

$\mathbb{R}^{d\times m}$: set of all $d \times m$ real matrices.

$\mathbb{P}$: Probability measure.

$\mathbb{E}$: Expectation.

$||f||^2_{a,b}$: $\mathbb{E}\left(\int_a^b |f(t)|^2 \,\mathrm{d}t\right)$.

$\mathscr{M}^2\left([a,b] : \mathbb{R}\right)$: $\left\{f = \{f_t\}_{a\leq t\leq b} \ : \ ||f||^2_{a,b} < \infty\right\}$.

$||\phi||$: $\sup_{t\in \mathrm{Dom}(\phi)} |\phi(t)|$.

$|x|$: $\sqrt{x^T \cdot x}$.

$\{\epsilon_n\}_{n\geq 0}$: Sequence of independent and identically distributed N(0,1) random variables.

sdde : Stochastic delay differential equations.

sdide : Stochastic delay integro differential equations.

MOTIVATION AND OBJECTIVES

Efficiency of market is an important hypothesis presumed often by researchers working in mathematical finance. It implies that market is perfectly competitive and information is accessible to everyone in no time and their response is also instantaneous. But from a pragmatic point of view this assumption is having little validity. In practice, a certain amount of time is required for the information to reach out to all stakeholders and responses to be decided by them. This simple but important observation captivated researchers across the world to incorporate a new parameter of time delay into the mathematical models under respective concern [23],[17]. But market inefficiency is not the only seminal reason for delay appearing in economic models. Memory involved in financial variables is also an equally pertinent reason. Conditional heteroskedastic models are actually based on this assumption that the future events occuring in financial market is a function of the past history. For instance consider the classical GARCH (1,1) (Generalized Auto Regressive Conditional Heteroskedasticty) scheme, in which,

$$\begin{cases} h_n = \epsilon_n \sigma_n \\ \sigma_n^2 = \alpha_0 + \alpha_1 h_{n-1}^2 + \beta_1 \sigma_{n-1}^2. \end{cases} \tag{1}$$

where $\{\epsilon_n\}_{n\geq 0}$ is a sequence of identically distributed independent (i.i.d) normal random variables with zero mean and unit variance. Inspired by the usefulness of this model and similarity it has with discretization of a differential equation people started to explore about the continuous time analogues of GARCH(1,1) scheme. As a result a large number of articles have been published in that direction ([5],[1],[7],[3],[6]). Yuriy Kazmerchuk et.al[15]. obtained a seemingly realistic result in this regard. He came by a non linear stochstic integro differential equation as a canonical limit of discrete GARCH process, which models of uncertainty in financial market. The aforementioned model is of the form,

$$\frac{\mathrm{d}\sigma^2(t,S_t)}{\mathrm{d}t} = \gamma V + \frac{\alpha}{\tau}\left[\int\limits_{t-\tau}^{t} \sigma(s,S_s)\,\mathrm{d}B(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t). \tag{2}$$

where $\sigma(t,S_t)$ is te volatility and $S_t = \{S(t+\theta) : -\tau \leq \theta \leq 0\}$. The above equation is driven by the squared Itô integral term, $\int\limits_{t-\tau}^{t} \sigma(s,S_s)\,\mathrm{d}B(s)$. Not only that, the function itself $\sigma^2(t, S_t)$ appear non linearly inside the stochastic integral.

One natural question we can pose is whether the equation (5.1) admits a solution or not. Clearly this equation can not be solved analytically. We have to use numerical methods to solve this equation to get approximate solutions. This gave us the initial impetus to study the properties of stochastic integro differential equations. As we proceeded, we found that the a large number of physical phenomena can be modeled using such equations, for instance population dyanamics, genetics and mathematical finance [12],[21].

Stochastic integro differential equations with *deterministic delay integral* terms was already surveyed in detail, for instance see [14],[24] and [13]. General theory of stochastic integro differential equations with deterministic and stochastic integral terms independent of delay is also a well studied subject [19],[20]. But the integro-differential equation we posses does not fit into the above models because of the intrinsic properties of the equation itself. It is explicitly driven by deterministic time variation (dt), and implicitly by a stochastic delay integral term. This motivated us to consider a new generic model of the form,

$$\mathrm{d}y(t) \;=\; F\left(t, y(t), Iy(t), Jy(t)\right)\,\mathrm{d}t + G\left(t, y(t), Iy(t), Jy(t)\right)\,\mathrm{d}B(t), \quad t \in [\tau, T], \tag{3}$$

where $Iy(t) = \int\limits_{t-\tau}^{t} f(s, y(s))\,\mathrm{d}s$ and $Jy(t) = \int\limits_{t-\tau}^{t} g(s, y(s))\,\mathrm{d}B(s)$. Here the positive constant τ is the delay in time. It is easy to check that equation (5.1) naturally come as a special case of the equation (3.1). We pose the following questions regarding the analytical and numerical properties of this equation,

1. Does the equation (3.1) has a solution and is it unique?
2. Is the solution bounded (in suitable norm) and is it stable ?
3. Can we numerically solve equation (3.1), and how does the error propagates as we time march ?
4. Are we able to solve the volatility equation in finance (5.1), using the tools we have developed ?

This thesis is a humble attempt to answer the above key questions. Despite the hardness and challenging nature of these problems, it highly inspiring and motivating to explore them, since the expansion of the field of quantitative finance is so vigorous and fast, making even the slightest achievements worthy enough to create great influence. We end this chapter by clearly specifying the objectives of this project.

Objectives

1) To establish a general model for stochastic delay integro-differential equations and study the following aspects.

   (i) Existence and uniqueness of solutions.

   (ii) Boundedness of solutions.

   (iii) Stability of solutions.

2. Derive a numerical method by which approximate solutions to a subclass of SDIDE can be obtained with sufficient accuracey.

   (i) Establish an error bound for the approximate solutions.

   (ii) Illustration by test equations.

3) Try to apply the tools developed to solve equation (5.1) which models volatility.

Chapter 1

Basic Stochastic Processes

1.1 Brownian Motion

Definition 1.1.1 (Brownian motion). Let $(\Omega, \mathbb{P}, \mathbb{F})$ be a probability space with a filtration $\{\mathbb{F}_t\}_{t\geq 0}$. A one dimensional Brownian motion is a real valued continuous $\{\mathbb{F}_t\}$ adapted process $\{B_t\}_{t\geq 0}$ with the following properties,

1) $B_0 = 0$,

2) for $0 \leq s < t < \infty$, the increment $B_t - B_s$, is normally distributed with zero mean and variance $t - s$.

3) for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of $\mathbb{F}_s$.

1.1.1 Existence of Brownian Motion

Lemma 1.1.2. *Let B(t) be a one dimensional Brownian motion. Then*

$$\mathbb{E}(B(t) \cdot B(s)) = min(t, s) \tag{1.1}$$

for any $t, s \in \mathbb{R}^+$

Proof. Assuming $t > s$,

$$\begin{aligned}
\mathbb{E}(B(t) \cdot B(s)) &= \mathbb{E}((B(t) - B(s) + B(s)) \cdot B(s)). \\
&= \mathbb{E}((B(t) - B(s)) \cdot B(s)) + \mathbb{E}((B(s))^2). \\
&= \mathbb{E}(B(t) - B(s)) \cdot \mathbb{E}(B(s)) + s. \\
&= s = \min(t, s).
\end{aligned}$$

The third equality followed from the fact that $B(t) - B(s)$ and $B(s)$ are independent of each other and $B(t) - B(s) \sim N(0, t-s)$, $B(s) \sim N(0, s)$. The other case where $s > t$ follows similarly. ☐

The existence of Brownian motion can be done in many ways. Here we give a comparatively simple method proved by Lévy and Ciesieslki [16].

Definition 1.1.3 (Haar functions). The family $\{h_k\}_{k\geq 1}$ of Haar functions are defined for $0 \leq t \leq 1$ as follows,

$$h_0(t) := 1 \quad \forall\, 0 \leq t \leq 1 \tag{1.2}$$

$$h_1(t) := \begin{cases} 1, & \forall\, 0 \leq t \leq \frac{1}{2} \\ -1, & \forall\, \frac{1}{2} < t \leq 1. \end{cases} \tag{1.3}$$

If $2^n < k < 2^{n+1}$ for $n = 1, 2, \ldots$ we define,

$$h_k(t) := \begin{cases} 2^{n/2}, & \frac{k-2^n}{2^n} \leq t \leq \frac{k-2^n+1/2}{2^n} \\ -2^{n/2}, & \frac{k-2^n+1/2}{2^n} < t \leq \frac{k-2^n+1}{2^n} \\ 0. & \text{otherwise.} \end{cases} \tag{1.4}$$

It can be proved that Haar functions $\{h_k\}_{k=0}^{\infty}$ forms a complete orthonormal basis of $L^2(0,1)$. We define the k^{th} Schauder function as,

$$s_k(t) := \int_0^t h_k(s)ds. \tag{1.5}$$

Theorem 1.1.4. *Let $\{A_k\}_{k\geq 0}$ be a sequence of N(0,1) random variables defined on the* *same probability space. Then the sum,*

$$W(t,\omega) := \sum_{k=0}^{\infty} A_k(\omega) S_k(t) \quad 0 \leq t \leq 1 \tag{1.6}$$

converges uniformly for all t, for a.e Ω. Further more,

1. *$W(\cdot)$ is a Brownian motion for $0 \leq t \leq 1$,*

2. *for a.e ω, the sample path $t \to W(\omega, t)$ is continuous.*

Please refer Introductory Stochastic Differential Equations by L.C. Evans [9] for the proof.

Theorem 1.1.5 (Existence of Browninan motion). *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be probability space on which countably many $N(0,1)$ i.i.d random variables $\{A_k\}_k^\infty$ are defined. Then there exists a Brownian motion for all $\omega \in \Omega$ and $t \in \mathbb{R}^+$.*

Proof of this theorem is given in *A simple contruction of certain diffusion processes*, by John Lamperti [16]. Please refer the above mentioned article for details.

Definition 1.1.6 (d-dimensional Brownian motion). An $\mathbb{R}^d$ valued stochastic process $B(\cdot) = (B_1(\cdot), B_2(\cdot), \ldots, B_d(\cdot))$ is called a d-dimensional Brownian motion if,

1) For each $k = 1, 2, \ldots, d$, $B_k(\cdot)$ is a one dimensional Brownian motion,

2) The σ algebras, $\mathscr{B}_k := \sigma(B_k(t), t \geq 0)$ are independent for $k = 1, 2, \ldots, d$.

1.2 Properties of Brownian Motion

The time realization for a fixed $\omega \in \Omega$ is called a path of a Brownian motion, $\{B(t)\}$. Regarding paths Brownian motion has two important properties; it is everywhere continuous but nowhere differentiable.

Definition 1.2.1 (Uniformly Hölder continuous function). Let $0 < \gamma \leq 1$. A function, $f : [0,T] \to \mathbb{R}$ is called uniformly Hölder continuous with exponent $\gamma > 0$ if there exists a positive constant C_γ such that for every $s, t \in [0,T]$,

$$|f(s) - f(t)| \leq C_\gamma |t-s|^\gamma. \tag{1.7}$$

Theorem 1.2.2 (Kolmogorov). *Let $S(.)$ be a stochastic process with continuous sample paths a.s, such that*

$$\mathbb{E}(|X(t) - X(s)|^{\beta}) \leq C|t - s|^{1+\alpha} \tag{1.8}$$

for constants $\alpha, \beta, C > 0$, and for $0 \leq s, t \leq T$. Then for each $0 < \gamma < \frac{\alpha}{\beta}$, $T > 0$ and almost ω, there exists a constant $K = K(\omega, \gamma, T)$ such that

$$|X(t, \omega) - X(s, \omega)| \leq K|t - s|^{\gamma} \tag{1.9}$$

for every $0 \leq s, t \leq T$. That is $X(t, \omega)$ is uniformly Hölder continuous for exponents $0 < \gamma < \frac{\alpha}{\beta}$.

Proposition 1.2.3. *If $B(\cdot)$ is an n-dimensional Brownian motion then, it is uniformly Hölder continuous for all exponents $0 < \gamma < \frac{1}{2}$.*

Proof. We have for $m = 1, 2 \ldots,$ and $r = |t - s|$

$$\begin{aligned}
\mathbb{E}(|B(t) - B(s)|^{2m}) &= \frac{1}{(2\pi r)^{n/2}} \int\limits_{\mathbb{R}^n} |x|^{2m} \exp \frac{-|x|^2}{2r} \, \mathrm{d}x. \\
&= \frac{1}{(2\pi r)^{n/2}} r^m \int\limits_{\mathbb{R}^n} |y|^{2m} \exp \frac{-|x|^2}{2} \, \mathrm{d}y. \\
&\leq Cr^m = C|t - s|^m.
\end{aligned}$$

Therefore the hypothesis of the theorem 1.2.2 hold with the constants $\beta = 2m$ and $\alpha = m - 1$. Thus we have $B(\cdot)$ is uniformly Hölder continuous for all exponents, $0 < \gamma < \frac{\alpha}{\beta} = \frac{1}{2} - \frac{1}{2m}$. This holds for all m. Thus we have, $0 < \gamma < \frac{1}{2}$. $\square$

We conclude this chapter by stating the results regarding the nowhere differentiability of Brownian motion and it's quadratic variation.

Theorem 1.2.4. *1) For each $\frac{1}{2} < \gamma < 1$ and almost every ω, $B(t, \omega)$ is nowhere Hölder continuous with exponent γ.*

2) In particular the sample paths, $t \to B(t, \omega)$ is nowhere differentiable for a.e $\omega \in \Omega$.

Please refer Dvortesky, Erdös and Kakutani [8] for a well drafted proof.

Definition 1.2.5. The quadratic variation of Brownian motion is defined $[B,B](t)$ is defined as

$$[B,B](t) := \lim \sum_{i=1}^{n} |B(t_i^n) - B(t_{i-1}^n)|^2$$

where the limit is taken over all shrinking partitions of [0,t] with $\delta_n = \max_i |t_{i+1} - t_i| \to 0$ as $n \to 0$.

Theorem 1.2.6. *Quadratic variation of a Brownian motion over [0,t] is t, but is of infinite variation in any interval [0,t] however small it is.*

1.3 Martingales

Definition 1.3.1. A stochastic process is called a martingale $\{M_t\}$ with respect to the filtration $\{\mathbb{F}_t\}$ and the probability measure $\mathbb{P}$ if,

1. $\mathbb{E}(|M_t|) < \infty$,
2. M_t is $\mathbb{F}_t$ measurable for each t,
3. $\mathbb{E}(M_t|\mathbb{F}_s) = M_s$ a.s, if $s < t$.

We list few important examples of martingales.

i) If $B(t)$ is a Brownian motion then $B(t), B(t) - t, \exp\left(uB(t) - t\frac{u^2}{2}\right)$ are martingales.

ii) For any integrable random variable X, $\mathbb{E}[X|\mathbb{F}_t]$ is a martingale.

iii) For any Poisson process $N(t)$ with intensity λ, $N(t) - \lambda$ (called the compensated Poisson process), $(N(t) - \lambda t)^2 - \lambda t$ and $\exp\left((\log(1-u)N(t) + u\lambda t\right)$ for $0 < u < 1$ are martingales.

In the definition 1.3.1, if $\mathbb{E}(M_t|\mathbb{F}_s) \leq M_s$ then the martingale is called a sub martingale and if the inequality is reversed is called a super martingale. We can easily show that a super martingale $M(t)$ on $0 \leq t \leq T$ is a martingale if and only if $\mathbb{E}(M(T)) = \mathbb{E}(M(0))$. We state the following properties of martingales without proof,

Theorem 1.3.2 (**Doob - Lévy martingale and uniform integrability**). *Let Y be a square integrable random variable then $M(t) = \mathbb{E}[Y|\mathbb{F}_t]$ is a uniformly integrable martingale. $M(t)$ is called* Doob - Lévy *martingale.*

Corollary 1.3.3. *Any martingale $M(t)$ on a finite time interval $0 \leq t \leq T < \infty$ is uniformly integrable and is closed by $M(T)$.*

Theorem 1.3.4 (**Martingale convergence theorem**). *If $M(t)$, $0 \leq t < \infty$ is a square integrable martingale, then there exists an almost sure limit $\lim_{t\to\infty} M(t) = Y$, and Y is an integrable random variable.*

Theorem 1.3.5 (**Doob**). *Let $\{X_n\}$ be a martingale, $\lim_{n\to\infty} E|X_n| < \infty$, then $\lim X_n = X$ almost surely.*

Theorem 1.3.6 (**Doob's martingale inequalities**).

1. *If $\{X_n\}$ is a square sub martingale indexed by the finite set $(0, 1, \dots, N)$, then for every $\lambda > 0$,*

$$\lambda \mathbb{P}\left[\sup_n X_n \geq \lambda\right] \leq \mathbb{E}[X_N \mathbb{I}_{\{\sup_n X_n \geq \lambda\}}] \leq \mathbb{E}[|X_N| \mathbb{I}_{\{\sup_n X_n \geq \lambda\}}] \tag{1.10}$$

2. *If X is a martingale or a positive sub martingale indexed by the finite set $(0, 1, \dots, N)$, then for every $p \geq 1$ and $\lambda > 0$,*

$$\lambda^p \times \mathbb{P}\left[\sup_n X_n \geq \lambda\right] \leq \mathbb{E}[|X_N|^p] \tag{1.11}$$

   and for $p > 1$,

$$\mathbb{E}[|X_N|^p] \leq \mathbb{E}[\sup_n |X_n|^p] \leq \left(\frac{p}{p-1}\right) \mathbb{E}[|X_N|^p] \tag{1.12}$$

Chapter 2

Stochastic Differential Equations

2.1 Stochastic Integrals

To maintain consistency with the notations and symbols, we have followed the conventions given in Stochastic Differential Equations with Applications, X. Mao [18]. Let $\{\Omega, \mathbb{F}, \mathbb{P}\}$ be a complete probability space that satisfies usual conditions. Let $B(t)$ be a one dimensional Brownian motion defined on this space adapted to the above filtration. we define the space $\mathscr{M}^2([a,b];\mathbb{R})$ for $0 \leq a < b < \infty$ as the set of real valued measurable $\{\mathbb{F}_t\}$ adapted processes $f = \{f_t\}_{a \leq t \leq b}$ such that

$$||f||_{a,b}^2 = \mathbb{E}\left(\int_a^b |f(t)|^2 \, \mathrm{d}t\right) < \infty. \tag{2.1}$$

Two elements $f_1, f_2 \in \mathscr{M}^2([a,b];R)$ are identified as the same, if $||f_1 - f_2||_{a,b}^2 = 0$, and we write $f_1 = f_2$.

Definition 2.1.1 (**Simple processes**). A real valued stochastic process $g = \{g(t)\}_{a \leq t \leq b}$ is called a simple process if there exists a partition $a = t_0 < t_1 < t_2 < \cdots < t_k = b$ of [a,b], and bounded random variables ξ_i, $0 \leq i \leq k-1$ such that ξ_i is $\mathbb{F}_{t_i}$ measurable and

$$g(t) = \xi_0 \mathbb{I}_{[t_0,t_1]}(t) + \sum_{i=1}^{k-1} \xi_i \mathbb{I}_{(t_i,t_{i+1}]}(t). \tag{2.2}$$

Denote by $\mathscr{M}_0([a,b];\mathbb{R})$ be the family of all such processes. Then we define

$$\int_a^b g(t)\,\mathrm{d}B(t) = \sum_{i=1}^{k-1} \xi_i (B_{i+1}(t) - B_i(t)) \tag{2.3}$$

and call it the stochastic integral of g with respect to the Brownian motion $\{B_t\}$ or the Itô integral.

The following results hold true,

Lemma 2.1.2. *If $g \in \mathscr{M}_0([a,b];\mathbb{R})$, then*

1. $\mathbb{E}\left(\int_a^b g(t)\,\mathrm{d}B(t)\right) = 0$

2. $\mathbb{E}\left|\int_a^b g(t)\,\mathrm{d}B(t)\right|^2 = \mathbb{E}\left(\int_a^b |g(t)|^2\,\mathrm{d}t\right)$

we define the general Itô integrals. as follows.

Theorem 2.1.3. *For any $f \in \mathscr{M}^2([a,b];\mathbb{R})$, there exists a sequence $\{g_n\}$ of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\left(\int_a^b |f(t) - g_n(t)|^2\,\mathrm{d}t\right) = 0 \tag{2.4}$$

Then the Itô integral of f with respect to $\{B_t\}$ is defined by

$$\int_a^b f(t)\,\mathrm{d}B(t) = \lim_{n\to\infty} \int_a^b g_n(t)\,\mathrm{d}B(t) \tag{2.5}$$

in $L^2(\Omega;\mathbb{R})$

2.1.1 Properties of Itô integrals

Let $f, g \in \mathscr{M}^2([a,b];\mathbb{R})$, and let α, β be two real numbers. Then we have

1. $\int_a^b f(t)\,\mathrm{d}B(t)$ is $\mathbb{F}_b$ measurable.

2. $\mathbb{E}\left(\int_a^b f(t)\,\mathrm{d}B(t)\right) = 0.$

3. $\mathbb{E}\left|\int_a^b f(t)\,\mathrm{d}B(t)\right|^2 = \mathbb{E}\left(\int_a^b |f(t)|^2\,\mathrm{d}t\right).$

4. $\int_a^b \alpha f(t) + \beta g(t)\,\mathrm{d}B(t) = \alpha \int_a^b f(t)\,\mathrm{d}B(t) + \beta \int_a^b g(t)\,\mathrm{d}B(t)$.

The following theorem is extremely useful, while estimating bounds when proving existence uniqueness theorems.

Theorem 2.1.4. *let $f \in \mathscr{M}^2([a,b];\mathbb{R})$, then the integral $I(t) = \int_0^t f(s)\,\mathrm{d}B(s)$ for $0 \leq t \leq T$ is a square integrable continuous martingale with respect to the filtration $\{\mathbb{F}_t\}$. In particular*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t f(s)\,\mathrm{d}B(s)\right|^2\right] \leq 4\mathbb{E}\left(\int_0^T |f(s)|^2\,\mathrm{d}s\right). \tag{2.6}$$

Definition 2.1.5. (**Quadratic variation of a martingale**) Let $M(t)$ be a square integrable continuous martingale. Then there exists a unique integrable adapted increasing process denoted by $[M,M]_t$ such that $\{M_t^2 - [M,M]_t\}$ is a continuous martingale vanishing at $t = 0$.

Theorem 2.1.6 (**Quadratic variation of martingale given in theorem 2.1.4**). *Let $f \in \mathscr{M}^2([a,b];R)$. Then the indefinite integral $\{I(t)\}$ defined in theorem 2.1.4 is square integrable continuous martingale with the quadratic variation given by,*

$$[I,I]_t = \int_a^b |f(s)|^2\,\mathrm{d}s \tag{2.7}$$

for $0 \leq t \leq T$

2.1.2 Moment Inequalities

Let $B(t) = (B(t_1), B(t_2), \ldots, B(t_d))^T$ be a d dimensional Brownian motion defined on the complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$ adapted to the filtration $\{\mathbb{F}_t\}_{t\geq 0}$. Then the following inequalities hold true.

1. Let $p \geq 2$ and $g \in \mathscr{M}^2([a,b];\mathbb{R}^{m\times d})$ such that

$$\mathbb{E}\left(\int_0^T |g(s)|^p \mathrm{d}s\right) < \infty.$$

Then

$$\mathbb{E}\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p \le \left(\frac{p(p-1)}{2}\right)^{p/2} T^{\frac{p-2}{2}}\mathbb{E}\left(\int_0^T |g(s)|^p\,\mathrm{d}s\right). \tag{2.8}$$

In particular if $p = 2$, there is equality. Moreover

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^T g(s)\,\mathrm{d}B(s)\right|^p\right) \le \left(\frac{p^3}{p(p-1)}\right)^{p/2} T^{\frac{p-2}{2}}\mathbb{E}\left(\int_0^T |g(s)|^p\,\mathrm{d}s\right). \tag{2.9}$$

2. Let $T > 0$ and $c \ge 0$. Let $u(\cdot)$ be a Borel measurable nonnegative function on [0,T], and let $v(\cdot)$ be nonnegative integrable function on [0,T]. If

$$u(t) \le c + \int_0^t v(s)u(s)ds \quad \text{for } 0 \le t \le T \tag{2.10}$$

then

$$u(t) \le c \times \exp\left(\int_0^t v(s)ds\right) \quad \text{for } 0 \le t \le T. \tag{2.11}$$

The above inequality (2) is called Gronwall's inequality. Next we state a class of lemmas called Itô's lemma which lies in the heart of stochastic differential equations.

2.2 Itô's Lemma

Definition 2.2.1. A one dimensional Itô process is a continuous adapted process $x(t)$ on $t \ge 0$ of the form

$$x(t) = x(0) + \int_0^t f(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}B(s) \tag{2.12}$$

where $f \in L^1(\mathbb{R}^+;\mathbb{R})$ and $g \in L^2(\mathbb{R}^+;\mathbb{R})$. We shall say $x(t)$ has the stochastic differential given by,

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}B(t). \tag{2.13}$$

Lemma 2.2.2 (One dimensional Itô's lemma). *Let $x(t)$ be an Itô process with the stochastic differential*

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}B(t) \tag{2.14}$$

where $f \in L^1(\mathbb{R}^+;\mathbb{R})$ *and* $g \in L^2(\mathbb{R}^+;\mathbb{R})$. *Let* $V \in C^{2,1}(\mathbb{R}\times\mathbb{R}^+;\mathbb{R})$. *Then* $V(x(t),t)$ *is again an Itô process with the stochastic differential,*

$$\begin{aligned} \mathrm{d}V(x(t),t) = \left[V_t(t,x(t)) + f(t)V_x(t,x(t)) + \frac{1}{2}g^2(t)V_{xx}(t,x(t))\right]\mathrm{d}t \\ + V_x(t,x(t))g(t)\mathrm{d}B_t. \end{aligned} \tag{2.15}$$

Definition 2.2.3. A d-dimensional Itô process is an $\mathbb{R}^d$ -valued continuous adapted process $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ on $t \geq 0$ of the form,

$$x(t) = x(0) + \int_0^t f(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}B(s) \tag{2.16}$$

where $f = (f_1, f_2, \ldots, f_d)^T \in L^1(\mathbb{R}^+;\mathbb{R}^d)$ and $g = (g_{ij})_{d\times m} \in L^2(\mathbb{R}^+;\mathbb{R}^{d\times m})$. we shall say that $x(t)$ has stochastic differential $\mathrm{d}x(t)$ on $t \geq 0$ given by

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}B(t). \tag{2.17}$$

Lemma 2.2.4 (Multi dimensional Itô's lemma). *Let* $x(t)$ *be a* d *dimensional Itô process on* $t \geq 0$ *with the stochastic differential*

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}B(t) \tag{2.18}$$

where $f = (f_1, f_2, \ldots, f_d)^T \in L^1(\mathbb{R}^+;\mathbb{R}^d)$ *and* $g = (g_{ij})_{d\times m} \in L^2(\mathbb{R}^+;\mathbb{R}^{d\times m})$. *Then* $V(x(t),t) \in \mathscr{C}^{2,1}(\mathbb{R}^d\times\mathbb{R}^+;\mathbb{R})$ *is again an Itô process with the stochastic differential given by*

$$\begin{aligned} \mathrm{d}V(x(t),t) = \left[V_t(t,x(t)) + V_x(t,x(t))f(t) + \frac{1}{2}\mathrm{Tr}\left(g^T(t)V_{xx}(x(t),t)g(t)\right)\right]\mathrm{d}t \\ + V_x(x(t),t)g(t)\mathrm{d}B(t). \end{aligned} \tag{2.19}$$

With this we stop the discussion on stochastic calculus and next chapter onwards we shall explain the real situation with which we are working. But any research in the following field require a thorough understanding of the previously mentioned two chapters.

Chapter 3

Stochastic delay integro differential equations

3.1 General description

Consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathbb{P}$ is the probability measure, with the filtration $\{\mathscr{F}_t\}_{t\geq 0}$. Let $\{B(t)\}_{t\geq 0}$ be a one dimensional Brownian motion adapted to the aforementioned filtration $\{\mathscr{F}\}_{t\geq 0}$. We propose to give a general framework for *stochastic (Itô type) integro differential equations involving delay integrals* to which equation becomes a genuine special case. In this chapter we consider generic stochastic integro differential equations with delay integrals of the form,

$$\begin{aligned} \mathrm{d}y(t) \;=\; & F\left(t, y(t), \int\limits_{t-\tau}^{t} f(s, y(s))\,\mathrm{d}s, \int\limits_{t-\tau}^{t} g(s, y(s))\,\mathrm{d}B(s)\right)\mathrm{d}t \\ & + G\left(t, y(t), \int\limits_{t-\tau}^{t} f(s, y(s))\,\mathrm{d}s, \int\limits_{t-\tau}^{t} g(s, y(s))\,\mathrm{d}B(s)\right)\mathrm{d}B(t). \end{aligned} \tag{3.1}$$

For simplicity, let us denote,

$$Iy(t) = \int\limits_{t-\tau}^{t} f(s, y(s))\,\mathrm{d}s,\;\; Jy(t) = \int\limits_{t-\tau}^{t} g(s, y(s))\,\mathrm{d}B(s).$$

We shall call the first integral, $Iy(t)$ as *deterministic delay integral* and the second integral $Jy(t)$ as *stochastic delay integral*. The properties of the coefficient functions will be delineated in a separated section. Here τ is a positive constant which represents delay in the time. From the definitions of $Iy(t)$ and $Jy(t)$, we can infer, the minimal initial data that must be specified to obtain a solution is $y(t) = \varphi(t)$ for $t \in [0, \tau]$. It is worthwhile to notice that the initial data is a member of some suitable function class. With properly defined coefficient functions, and the initial data which belongs suitable function space, the immediate question to answer is the existence of solutions for the equation (3.1) in a bounded interval $[\tau, T]$ for some $T > \tau$. Towards this goal, in the next section, we clearly specify the properties of coefficient functions appearing in the equation (3.1) and the assumptions we impose on them. We shall be using these conditions and hypotheses to prove many interesting results further ahead.

3.2 Assumptions

Firstly, we give the range and domain of the following functions,

$$F : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}; \quad G : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}. \tag{3.2}$$

$$f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}; \quad g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}. \tag{3.3}$$

In addition to that $F, G \in \mathbb{L}^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and $f, g \in \mathbb{L}^2(\mathbb{R}_+ \times \mathbb{R})$. Moreover F, G, f and g are assumed to be continuous and $\{\mathscr{F}_t\}$ adapted random functions. We assume that,

A1). $|F(t,x,y,z)|^2 + |G(t,x,y,z)|^2 \le K^2(1 + |x|^2 + |y|^2 + |z|^2)$,

A2). $|f(t,x)|^2 + |g(t,x)|^2 \le K^2(1 + |x|^2)$,

B1). $|F_i(t,x_1,y_1,z_1) - F_i(t,x_2,y_2,z_2)|^2 \le K^2(|x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2)$,

B2). $|f_i(t,x_1) - f_i(t,x_2)|^2 \le K^2(|x_1 - x_2|^2)$.

where $F_i = F$ or G and $f_i = f$ or g. Without loss of generality we can assume K as a global constant since assumptions $A1$ to $B2$ involve only finite number of constants, and we can choose the maximum of them as K. We transform equation (3.1) to the integral form,

$$y(t) = \varphi(\tau) + \int_\tau^t F(s, y(s), Iy(s), Jy(s))\,\mathrm{d}s + \int_\tau^t G(s, y(s), Iy(s), Jy(s))\,\mathrm{d}B(s). \tag{3.4}$$

Finally, we assume that $\varphi(t) \in \mathbb{L}^2([0,\tau])$, continuous and $\{\mathscr{F}_t\}$ adapted.

3.3 Main Theorem

We intend by this section to establish existence and uniqueness of the solution for equation (3.1). The traditional method of successive approximations is used to obtain solutions. We define the successive approximates as,

$$\left.\begin{aligned} y_n(t) &= \varphi(t)\mathbb{I}_{[0,\tau]} + \varphi(\tau)\mathbb{I}_{(\tau,T]} + \int_\tau^t \mathbb{I}_{(\tau,T]}(s)F(s, y_{n-1}(s), Iy_{n-1}(s), Jy_{n-1}(s))\,\mathrm{d}s. \\ &\qquad + \int_\tau^t \mathbb{I}_{(\tau,T]}(s)G(s, y_{n-1}(s), Iy_{n-1}(s), Jy_{n-1}(s))\,\mathrm{d}B(s) \\ y_0(t) &= \varphi(t)\mathbb{I}_{[0,\tau]} + \varphi(\tau)\mathbb{I}_{(\tau,T]}. \end{aligned}\right\} \tag{3.5}$$

For $t \in [\tau, T]$, we can express the above expression in a simplified form as,

$$\begin{aligned} y_{n+1}(t) = \varphi(\tau) &+ \int_\tau^t F(s, y_n(s), Iy_n(s), Jy_n(s))\,\mathrm{d}s \\ &+ \int_\tau^t G(s, y_n(s), Iy_n(s), Jy_n(s))\,\mathrm{d}B(s). \end{aligned} \tag{3.6}$$

Together with the previously defined successive approximates, we can proceed to the existence and uniqueness of solutions.

Theorem 3.3.1 (Existence and Uniqueness)**.** *Assume $A1, A2, B1$ and $B2$ hold true. If $\varphi(t)$ is an $\mathscr{F}_t$ adapted continuous process with $\sup_{0\le t\le\tau}|\varphi(t)|^2 < \infty$, then there exists a unique solution $y(t)$ for equation (3.4).*

Proof. We have,

$$\begin{aligned} y_{n+1}(t) - y_n(t) &= \int_\tau^t (F(s, y_n(s), Iy_n(s), Jy_n(s)) - F(s, y_{n-1}(s), Iy_{n-1}(s), Jy_{n-1}(s)))\mathrm{d}s \\ &+ \int_\tau^t (G(s, y_n(s), Iy_n(s), Jy_n(s)) - G(s, y_{n-1}(s), Iy_{n-1}(s), Jy_{n-1}(s)))\mathrm{d}B(s). \end{aligned} \tag{3.7}$$

The following estimate is straightforward using Schwartz inequality, isometric property of Itô integral, and assumptions $A1$ and $B1$.

$$\begin{aligned}
\mathbb{E}\left(|y_1(t)-y_0(t)|^2\right) &\le 2K^2(1+t-\tau)\mathbb{E}\int_\tau^t \left(1+|y_0(s)|^2+|Iy_0(s)|^2+|Jy_0(s)|^2\right)\,\mathrm{d}s \\
&\le 2K^2(t-\tau)(1+t-\tau)(1+K^2\tau^2+K^2\tau)\left[1+\mathbb{E}\sup_{0\le t\le\tau}|\varphi(t)|^2\right] \\
&\le \mathcal{M}_2\mu(t-\tau). \qquad (3.8)
\end{aligned}$$

where $\mathcal{M}_2(t) = 2K^2(1+t-\tau)p(\tau)$, $\mu = 1+\mathbb{E}\left(\sup_{0\le t\le\tau}|\varphi(t)|^2\right)$, and $p(x) = 1+K^2x+K^2x^2$. Similarly,

$$\begin{aligned}
\mathbb{E}|y_{n+1}(t)-y_n(t)|^2 &\le 2K^2(t-\tau+1)\int_\tau^t \mathbb{E}|y_n(s)-y_{n-1}(s)|^2\,\mathrm{d}s \\
&+ 2K^2(t-\tau+1)\int_\tau^t \mathbb{E}|Iy_n(s)-Iy_{n-1}(s)|^2\,\mathrm{d}s \;+\; 2K^2(t-\tau+1)\int_\tau^t \mathbb{E}|Jy_n(s)-Jy_{n-1}(s)|^2\,\mathrm{d}s. \qquad (3.9)
\end{aligned}$$

The last two terms in equation (3.9) can be estimated as,

$$\int_\tau^t \mathbb{E}|Iy_n(s)-Iy_{n-1}(s)|^2\,\mathrm{d}s \le K^2\tau(t-\tau)\int_\tau^t |y_n(s)-y_{n-1}(s)|^2\,\mathrm{d}s.$$

$$\int_\tau^t \mathbb{E}|Jy_n(s)-Jy_{n-1}(s)|^2\,\mathrm{d}s \le K^2(t-\tau)\int_\tau^t |y_n(s)-y_{n-1}(s)|^2\,\mathrm{d}s.$$

Substituting these bounds on equation (3.9),

$$\begin{aligned}
\mathbb{E}|y_{n+1}(t)-y_n(t)|^2 &\le 2K^2(t-\tau+1)(1+K^2\tau(t-\tau)+K^2(t-\tau))\int_\tau^t \mathbb{E}|y_n(s)-y_{n-1}(s)|\,\mathrm{d}s \\
&\le 2K^2(t-\tau+1)(1+p(t)-p(\tau))\int_\tau^t \mathbb{E}|y_n(s)-y_{n-1}(s)|\,\mathrm{d}s \\
&\le \mathcal{M}_2(t)\int_\tau^t \mathbb{E}|y_n(s)-y_{n-1}(s)|\,\mathrm{d}s. \qquad (3.10)
\end{aligned}$$

without any confusion we redefine $\mathcal{M}_2(t) = 2K^2(4+t-\tau)\max\left(p(\tau), 1+p(t)-p(\tau)\right)$, so that both equations (3.8) and (3.10) hold true. Finally assume that

$$\mathbb{E}|y_{n+1}(t)-y_n(t)|^2 \le \frac{(\mathcal{M}_2(t)(t-\tau))^{n+1}}{(n+1)!}\mu. \qquad (3.11)$$

. We proceed by induction to prove the assumption,

$$\begin{aligned}\mathbb{E}|y_{n+2}(t)-y_{n+1}(t)|^2 &\le \mathcal{M}_2(t)\int_\tau^t \mathbb{E}|y_{n+1}(s)-y_n(s)|\,\mathrm{d}s\\ &\le \mathcal{M}_2(t)\int_\tau^t \frac{(\mathcal{M}_2(s)(s-\tau))^{n+1}}{(n+1)!}\mu\,\mathrm{d}s\\ &= \frac{(\mathcal{M}_2(t)(t-\tau))^{n+2}}{(n+2)!}. \end{aligned} \tag{3.12}$$

In the preceding inequality, we have used the monotonicity of the bound function $\mathcal{M}_2(t)$. Then again using Schwartz inequality, isometry property and Doob's martingale inequality we can prove that,

$$\begin{aligned}\mathbb{E}\left(\sup_{\tau\le t\le T}|y_{n+1}(t)-y_n(t)|^2\right) &\le 2K^2(4+T-\tau)(1+p(T)-p(\tau))\int_\tau^T \mathbb{E}|y_n(s)-y_{n-1}(s)|^2\,\mathrm{d}s\\ &= \mathcal{M}_2(T)\int_\tau^T \mathbb{E}|y_n(s)-y_{n-1}(s)|^2\,\mathrm{d}s\\ &\le \frac{(\mathcal{M}_2(T)(T-\tau))^{n+1}}{(n+1)!}\mu.\end{aligned}$$

Applying Chebyshev's inequality,

$$\mathbb{P}\left(\sup_{\tau\le t\le T}|y_{n+1}(t)-y_n(t)|^2 \ge 2^{-n}\right) \le \frac{(2\mathcal{M}_2(T)(T-\tau))^{n+1}}{(n+1)!}\mu. \tag{3.13}$$

Since $\displaystyle\sum_n 2^{-n}\frac{(2\mathcal{M}_2(T)(T-\tau))^n}{n!}\mu \le \infty$, Borel - Cantelli lemma implies,

$$\mathbb{P}\left(\sup_{\tau\le t\le T}|y_{n+1}(t)-y_n(t)|^2 < 2^{-n}, n\uparrow\infty\right) = 1. \tag{3.14}$$

Therefore the partial sums, $\sum_{i=0}^{n-1} y_{i+1}(t)-y_i(t)+\varphi(\tau)=y_n(t)$ is uniformly convergent in $t\in[\tau,T]$ as well as in $\mathbb{L}^2$ norm by (3.12). Denote this uniform limit by $y(t)$ and because of being the uniform limit of $\mathscr{F}_t$ adapted continuous functions, $y(t)$ is also continuous and $\mathscr{F}_t$ adapted.

Next we check the uniqueness. For that purpose assume that $x(t)$ and $y(t)$ are two solutions of (3.1), then as we did in the existence part for some generic constant $C(T)$,

$$\mathbb{E}\left(\sup_{\tau\le s\le t}|x(s)-y(s)|^2\right) \le 2(4+t-\tau)C(T)\int_\tau^t \sup_{\tau\le u\le s}|x(u)-y(u)|^2\,\mathrm{d}s. \tag{3.15}$$

Then Gronwall's inequality readily implies,

$$\mathbb{E}\left(\sup_{\tau\le s\le t}|x(s)-y(s)|^2\right)=0. \tag{3.16}$$

Therefore $x(t)=y(t)$ for every $t\in[\tau,T]$ in the almost everywhere sense. We conclude the theorem by establishing $x(t)$ indeed satisfies equation (3.4). For this observe that,

$$\begin{aligned}\mathbb{E}\Bigg(\bigg|\int_\tau^t &(F(s,y_n(s),Iy_n(s),Jy_n(s))-F(s,y(s),Iy(s),Jy(s))\,\mathrm{d}s\\ +\int_\tau^t &(G(s,y_n(s),Iy_n(s),Jy_n(s))-G(s,y(s),Iy(s),Jy(s)))\,\mathrm{d}B(s)\bigg|^2\Bigg)\\ &\le 2(1+t-\tau)\tilde{C}(T)\int_t^t\mathbb{E}|x_n(s)-x(s)|^2\,\mathrm{d}s. \end{aligned}\tag{3.17}$$

Since $y_n(t)\to y(t)$ for every $t\in[\tau,T]$ in $\mathbb{L}^2$, R.H.S converges to 0 in $\mathbb{L}^2$ norm. Hence by dominated convergence theorem (please see L.C. Evans [10]),

$$\begin{aligned}&\int_\tau^t F(s,y_n(s),Iy_n(s),Jy_n(s))\,\mathrm{d}s+\int_\tau^t G(s,y_n(s),Iy_n(s),Jy_n(s))\,\mathrm{d}B(s)\\ &\to\int_\tau^t F(s,y(s),Iy(s),Jy(s))\,\mathrm{d}s+\int_\tau^t G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s)\end{aligned}$$

in $\mathbb{L}^2$. Therefore, passing limit $n\to\infty$ in equation (3.6) we obtain the desired result.

This completes the proof of the entire theorem. □

Though existence and uniqueness is proved, it is equally important to indicate the space in which the solutions lives. This forms the objective of the next theorem. We show that $y(t)\in\mathscr{M}^2([0,T];\mathbb{R})$.

Proposition 3.3.2. *Under the assumptions of theorem 3.3.1, solution of the equation* (3.1) $y(t)\in\mathscr{M}_2([0,T];\mathbb{R})$.

Proof. We have

$$|y(t)|^2\le\ 3|\varphi(\tau)|^2+3\left|\int_\tau^t F(s,y(s),Iy(s),Jy(s))\,\mathrm{d}s\right|^2+3\left|\int_\tau^t G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s)\right|^2,$$

where we used the inequality $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$. Therefore,

$$\sup_{0\leq s\leq t} |y(t)|^2 \leq 3|\varphi(\tau)|^2 + \quad 3(t-\tau)\sup_{0\leq s\leq t}\int_\tau^s |F(s,y(s),Iy(s),Jy(s)|^2\,\mathrm{d}s$$
$$+3\sup_{0\leq s\leq t}\left|\int_\tau^s G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s)\right|^2.$$

For obtaining the preceding inequality, we have used Schwartz inequality. Taking expectation on both sides and applying Doob's martingale inequality we shall obtain,

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq t}|y(s)|^2\right) &\leq 3\left(\mathbb{E}|\varphi(\tau)|^2 + (t-\tau)\int_\tau^t \mathbb{E}|F(s,y(s),Iy(s),Jy(s))|^2\,\mathrm{d}s\right.\\ &\quad\left. + 4\int_\tau^t \mathbb{E}|G(s,y(s),Iy(s),Jy(s))|^2\,\mathrm{d}s\right)\\ &\leq 3\left(\mathbb{E}|\varphi(\tau)|^2 + (4+t-\tau)K^2p(t-\tau)\int_\tau^t\left(1+\mathbb{E}\sup_{0\leq u\leq s}|y(u)|^2\right)\mathrm{d}s\right).\end{aligned}$$

Therefore by Gronwall's inequality,

$$\mathbb{E}\left(\sup_{0\leq s\leq T}|y(s)|^2\right) \leq 3\mathbb{E}|\varphi(\tau)|^2\exp\left(3(4+T-\tau)(T-\tau)p(T-\tau)\right)-1<\infty. \tag{3.18}$$

Therefore,

$$y(t)\in\mathscr{M}^2\left([0,T]:\mathbb{R}\right).$$

□

3.4 Boundedness of the solution

Since we have a global bound on the solution $y(t)$, given by (3.18), it is natural to explore weather the p^{th} moment of the solution is bounded or not, as a routine step in stochastic calculus. We shall derive an exponential estimate for the p^{th} moment of the function. Structure of the proof is same as that of the derivation of exponential estimates of the p^{th} moment of solution of a general stochastic differential equation (reader may please see [18])

Theorem 3.4.1 (p^{th} moment exponential estimate)**.** *Let $p\geq 2$, and suppose that as-*

sumptions A2 and B2 hold true. Then

$$\mathbb{E}\left(\sup_{0\le s\le T}|y(s)|^2\right)\le 2^{\frac{p}{2}}(1+\mathbb{E}|\phi(\tau)|^2)\times\exp\left(\alpha(T-\tau)\right), \tag{3.19}$$

where $\alpha=\dfrac{p(33pK^2+(8-K)K)}{2}\left(1+K^2\tau^2+K^2\tau\right).$

Proof. Applying Itô formula to the function $(1+|y(t)|^2)^{\dfrac{p}{2}}$, we shall obtain,

$$\begin{aligned}
\mathrm{d}(1+|y(t)|^2)^{\dfrac{p}{2}} &= \left[\frac{p}{2}(1+|y(t)|^2)^{\dfrac{p-2}{2}}\times 2|y(t)|F(t,y(t),Iy(t),Jy(t))\right.\\
&+\left.\frac{1}{2}G^2(t,y(t),Iy(t),Jy(t))\left(p(1+|y(t)|^2)^{\dfrac{p-2}{2}}+|y(t)|\times p(p-2)(1+|y(t)|^2)^{\dfrac{p-4}{2}}|y(t)|\right)\right]\mathrm{d}t+\\
&+\left[\frac{p}{2}(1+|y(t)|^2)^{\dfrac{p-2}{2}}\times 2|y(y)|\times G(t,y(t),Iy(t),Jy(t))\right]\mathrm{d}B(t)\\
&=\left[p(1+|y(t)|^2)^{\dfrac{p-2}{2}}|y(t)|F(t,y(t),Iy(t),Jy(t))+\frac{p}{2}G^2(t,y(t),Iy(t),Jy(t))(1+|y(t)|^2)^{\dfrac{p-2}{2}}\right.\\
&+\left.\frac{p(p-2)}{2}\left(G(t,y(t),Iy(t),Jy(t))y(t)\right)^2(1+|y(t)|^2)^{\dfrac{p-4}{2}}\right]\mathrm{d}t\\
&+p(1+|y(t)|^2)^{\dfrac{p-2}{2}}|y(t)|G(t,y(t),Iy(t),Jy(t))\,\mathrm{d}B(t).
\end{aligned}$$

Integrating the above expression from τ to t, we shall obtain,

$$\begin{aligned}
(1+|y(t)|^2)^{\dfrac{p}{2}} &\le 2^{\frac{p-2}{2}}(1+|\varphi(\tau)|^p)+p\int_\tau^t(1+|y(s)|^2)^{\dfrac{p-2}{2}}y(s)F(s,y(s),Iy(s),Jy(s))\mathrm{d}s\\
&+\int_\tau^t(1+|y(s)|^2)^{\dfrac{p-2}{2}}G^2(s,y(s),Iy(s),Jy(s))\mathrm{d}s\\
&+\frac{p(p-2)}{2}\int_\tau^t(1+|y(s)|^2)^{\dfrac{p-4}{2}}(1+|y(s)|^2)G^2(s,y(s),Iy(s),Jy(s))\mathrm{d}s\\
&+p\int_\tau^t(1+|y(s)|^2)^{\dfrac{p-2}{2}}y(s)G(s,y(s),Iy(s),Jys(s))\mathrm{d}B(s).
\end{aligned}$$

Next, we club together second and third integral in the above expression. After that, upon rearranging the terms, we shall obtain,

$$
\begin{aligned}
(1+|y(t)|^2)^{\frac{p}{2}} \leq\ & 2^{\frac{p-2}{2}}(1+|\varphi(\tau)|^2) \\
& + p\int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} \left(y(s)F(s,y(s),Iy(s),Jy(s)) + \frac{p-1}{2} G^2(s,y(s),Iy(s),Jy(s)) \right) \mathrm{d}s \\
& + p\int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} y(s)G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s).
\end{aligned}
$$

Then by applying linear growth condition that has been specified in the initial section we obtain,

$$
\begin{aligned}
(1+|y(t)|^2)^{\frac{p-2}{2}} \leq\ & 2^{\frac{p-2}{2}}(1+|\varphi(\tau)|^2) \\
& + p\int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} \left(4K + \frac{K^2(p-1)}{2} \right) (1+|y(s)|^2 + |Iy(s)|^2 + |Jy(s)|^2)\mathrm{d}s \\
& + p\int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} y(s)G(s,y(s),Iy(s),Jy(s))\mathrm{d}B(s). \\
\leq\ & 2^{\frac{p-2}{2}}(1+|\varphi(\tau)|^2) + p\left(4K + \frac{K^2(p-1)}{2} \right) \left[\int_\tau^t (1+|y(s)|^2)^{\frac{p}{2}} \mathrm{d}s + \right. \\
& \left. + \int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} |Iy(s)|^2 \mathrm{d}s + \int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} |Jy(s)|^2 \mathrm{d}s \right] + \\
& p\int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} y(s)G(s,y(s),Iy(s),Jy(s))\mathrm{d}B(s).
\end{aligned}
$$

We have to estimate each of the integrals in the first square bracket. On estimating them,

$$
\int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} |Iy(s)|^2 \mathrm{d}s \leq K^2\tau^2 \int_\tau^t \left(1 + \sup_{[s-\tau,s]} |x(u)|^2 \right)^{\frac{p}{2}} \mathrm{d}s. \tag{3.20}
$$

Similarly,

$$
\int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} |Jy(s)|^2 \mathrm{d}s \leq \int_\tau^t (1+|y(s)|^2)^{\frac{p-2}{2}} \left| \int_{s-\tau}^s g(u,y(u))\mathrm{d}B(u) \right|^2 \mathrm{d}s. \tag{3.21}
$$

Therefore,

$$(1+|y(s)|^2)^{\dfrac{p}{2}} \leq 2^{\frac{p-2}{2}}(1+|\varphi(\tau)|^2) + p\left(4K + \frac{K^2(p-1)}{2}\right)\left[\int\limits_{\tau}^{t}(1+|y(s)|^2)^{\dfrac{p}{2}}\mathrm{d}s\right.$$

$$\left. + K^2\tau^2\int\limits_{\tau}^{t}\left(1+\sup_{[s-\tau,s]}|y(u)|^2\right)^{\dfrac{p}{2}}\mathrm{d}s + \int\limits_{\tau}^{t}(1+|y(s)|^2)^{\dfrac{p-2}{2}}\left|\int\limits_{s-\tau}^{s}g(u,y(u))\mathrm{d}B(u)\right|^2\mathrm{d}s\right]$$

$$+ p\int\limits_{\tau}^{t}(1+|y(s)|^2)^{\dfrac{p-2}{2}}y(s)G(s,y(s),Iy(s),Jy(s))\mathrm{d}B(s). \tag{3.22}$$

Taking expectation of supremum on both sides we shall obtain,

$$\mathbb{E}\left(\sup_{0\leq s\leq t}(1+|y(s)|^2)^{\dfrac{p}{2}}\right) \leq 2^{\frac{p-2}{2}}(1+\mathbb{E}|\phi(\tau)|^2)$$

$$+ \ p\left(4K+\frac{K^2(p-1)}{2}\right)\left[\int\limits_{\tau}^{t}\mathbb{E}\left(1+\sup_{0\leq u\leq s}|y(u)|^2\right)^{\dfrac{p}{2}}\mathrm{d}s + K^2\tau^2\int\limits_{\tau}^{t}\mathbb{E}\left(1+\sup_{s-\tau\leq u\leq s}|y(u)|^2\right)^{\dfrac{p}{2}}\mathrm{d}s\right.$$

$$\left. + \ K^2\tau\int\limits_{\tau}^{t}\mathbb{E}\left(1+\sup_{s-\tau\leq u\leq s}|y(u)|^2\right)^{\dfrac{p}{2}}\mathrm{d}s\right]$$

$$+ \ p\mathbb{E}\left(\sup_{0\leq s\leq t}\int\limits_{\tau}^{s}(1+|y(s)|^2)^{\dfrac{p-2}{2}}y(s)G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s)\right). \tag{3.23}$$

Using equation (2.9) we can estimate the last term as follows,

$$p\mathbb{E}\left(\sup_{0\leq s\leq t}\int\limits_{\tau}^{s}(1+|y(s)|^2)^{\dfrac{p-2}{2}}y(s)G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s)\right)$$

$$\leq p\mathbb{E}\left(\sup_{0\leq s\leq t}\left|\int\limits_{\tau}^{s}(1+|y(s)|^2)^{\dfrac{p-2}{2}}y(s)G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s)\right|\right)$$

$$\leq p\sqrt{\left(\frac{32}{1}\right)}\mathbb{E}\left(\int\limits_{\tau}^{t}(1+|y(s)|^2)^{p-2}|y(s)G(s,y(s),Iy(s),Jy(s)|^2\mathrm{d}s\right)^{\dfrac{1}{2}}$$

$$\leq 4p\sqrt{2}\mathbb{E}\left(\sup_{\tau\leq s\leq t}(1+|y(s)|^2)^{\frac{p}{2}}\int_{\tau}^{t}(1+|y(s)|^2)^{\frac{p-4}{2}}|y(s)G(s,y(s),Iy(s),Jy(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\mathbb{E}\left(\sup_{0\leq s\leq t}(1+|y(s)|^2)^{\frac{p}{2}}\right)+16p^2K^2\mathbb{E}\left(\int_{\tau}^{t}(1+|y(s)|^2)^{\frac{p-2}{2}}(1+|y(s)|^2+|Iy(s)|^2+|Jy(s)|^2)\,\mathrm{d}s\right).$$

Then as done in the estimate (3.23), we shall have,

$$p\mathbb{E}\left(\sup_{0\leq s\leq t}\int_{\tau}^{s}(1+|y(s)|^2)^{\frac{p-2}{2}}y(s)G(s,y(s),Iy(s),Jy(s))\,\mathrm{d}B(s)\right)\leq$$

$$\frac{1}{2}\mathbb{E}\left(\sup_{0\leq s\leq t}(1+|y(s)|^2)^{\frac{p}{2}}\right)+16p^2K^2\left[\int_{\tau}^{t}\mathbb{E}\left(1+\sup_{0\leq u\leq s}|y(u)|^2\right)^{\frac{p}{2}}\mathrm{d}s\right.$$

$$\left.+K^2\tau^2\int_{\tau}^{t}\mathbb{E}\left(1+\sup_{s-\tau\leq u\leq s}|y(u)|^2\right)^{\frac{p}{2}}\mathrm{d}s+K^2\tau\int_{\tau}^{t}\mathbb{E}\left(1+\sup_{s-\tau\leq u\leq s}|y(u)|^2\right)^{\frac{p}{2}}\mathrm{d}s\right].\tag{3.24}$$

Combining the inequalities (3.24) and (3.23) we will come by the following inequality,

$$\frac{1}{2}\mathbb{E}\left((1+\sup_{0\leq s\leq t}|y(s)|^2)^{\frac{p}{2}}\right)\leq 2^{\frac{p-2}{2}}(1+\mathbb{E}|\phi(\tau)|^2)$$

$$+\ \frac{p(33pK^2+(8-K)K)}{2}\left[\int_{\tau}^{t}\mathbb{E}\left(1+\sup_{0\leq u\leq s}|y(u)|^2\right)^{\frac{p}{2}}\mathrm{d}s+\ K^2\tau^2\int_{\tau}^{t}\mathbb{E}\left(1+\sup_{0\leq u\leq s}|y(u)|^2\right)^{\frac{p}{2}}\mathrm{d}s\right.$$

$$\left.+K^2\tau\int_{\tau}^{t}\mathbb{E}\left(1+\sup_{0\leq u\leq s}|y(u)|^2\right)^{\frac{p}{2}}\mathrm{d}s\right]\tag{3.25}$$

$$\mathbb{E}\left(\sup_{0\leq s\leq t}(1+|y(s)|^2)^{\frac{p}{2}}\right)\leq 2^{\frac{p}{2}}(1+\mathbb{E}|\phi(\tau)|^2)$$

$$+\ \frac{p(33pK^2+(8-K)K)}{2}\left(1+K^2\tau^2+K^2\tau\right)\int_{\tau}^{t}\mathbb{E}\left(1+\sup_{0\leq u\leq s}|y(u)|^2\right)^{\frac{p}{2}}\mathrm{d}s\tag{3.26}$$

Applying Gronwall's inequality we shall obtain,

$$\mathbb{E}\left((1+\sup_{0\le s\le t}|y(s)|^2)^{\dfrac{p}{2}}\right)\le$$
$$2^{\frac{p}{2}}(1+\mathbb{E}|\phi(\tau)|^2)\times\exp\left(\frac{p(33pK^2+(8-K)K)}{2}\left(1+K^2\tau^2+K^2\tau\right)(t-\tau)\right). \tag{3.27}$$

Therefore,

$$\mathbb{E}\left(\sup_{0\le s\le t}|y(s)|^2\right)\le 2^{\frac{p}{2}}(1+\mathbb{E}|\phi(\tau)|^2)\times\exp\left(\frac{p(33pK^2+(8-K)K)}{2}\left(1+K^2\tau^2+K^2\tau\right)(t-\tau)\right). \tag{3.28}$$

Hence,

$$\|y(s)\|_{0,T}^2\le 2^{\frac{p}{2}}(1+\mathbb{E}|\phi(\tau)|^2)\times\exp\left(\frac{p(33pK^2+(8-K)K)}{2}\left(1+K^2\tau^2+K^2\tau\right)(T-\tau)\right).$$

This completes the proof. □

3.5 Stability of solutions

In order to analyze the stability of solutions, we restrict our attention to a subclass of the general equation (3.1). We consider stochastic delay integro differential equation in which the delay integral and $y(t)$ appear as arguments of separate function. Firstly, we define the general form of the equation as,

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \gamma y(t)+f\left(t,\displaystyle\int_{t-\tau}^{t} g(s,y(s))\,\mathrm{d}B(s)\right) \quad \text{for } t\in[\tau,T], \\ y(t)=\varphi(t) \quad \text{for } t\in[0,\tau]. \end{cases} \tag{3.29}$$

where $\gamma\in\mathbb{R}$. Before delving into the analysis we must justify our selection,

- Stability of (3.29) can be studied easily because of its simple formulation. We can extend results that was being done in the deterministic delay differential equation to the equations of the form (3.29).
- (3.29) is of sufficient generality by which volatility equation becomes a particular

case of it.

Theory of integro differential equation with deterministic delay is well studied. For instance, consider the following type of equations.

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = f\left(t, y(t), \displaystyle\int_{t-\tau}^{t} g(s, y(s))\,\mathrm{d}s\right) \quad \text{for } t \in [\tau, T], \\ y(t) = \varphi(t) \quad \text{for } t \in [0, \tau]. \end{cases} \tag{3.30}$$

Properties of the above equation has been thoroughly investigated by different authors. For instance Chengjian Zhang et al. [25] studied the stability of the (3.30) rigorously. We can extend these results to equation (3.29) easily. Reader may please refer [25] for further details. We make use of the following definitions.

Definition 3.5.1. The stochastic system (3.29) is said to be globally stable in second moment if there exists a constant C such that,

$$\mathbb{E}\left(y(t) - \tilde{y}(t)\right)^2 \leq C \times \sup_{0 \leq \theta \leq \tau} \mathbb{E}(\varphi(\theta) - \psi(\theta))^2, \tag{3.31}$$

for every $t \in [\tau, T]$, where $y(t)$ and $\tilde{y}(t)$ are solutions of (3.29) with respective data $\varphi(t)$ and $\psi(t)$.

Before stating the main theorem we shall state a lemma by C.Zhang et.al. [25]. It is an extension of Halanay's theorem, which was proven by Baker and Tang [4].

Lemma 3.5.2 (Halanay)**.** *Assume that the scalar function $v(t)$ is continuous and non negative for $t \geq t_0 - \tau$, and satisfies,*

$$D_+ v(t) \leq -Av(t) + B \sup_{t-\tau \leq s \leq t} v(s), \quad \forall t \in [t_0, \infty) \tag{3.32}$$

and $v(t) = |\Phi(t)|$ for every $t \in [t_0 - \tau, t_0]$, wher $D_+ v(t)$ is the right hand derivative of $v(t)$ and $\Phi(t)$ is continuous and not identically vanishing for $t \in [t_0 - \tau, t_0]$. A and B are non negative constants with $-A + B < 0$. Then,

$$v(t) \leq \sup_{t_0 - \tau \leq s \leq t_0} |\Phi(s)|, \quad \forall t \geq t_0 \tag{3.33}$$

The following theorem is an easy consequence of the above lemma.

Theorem 3.5.3 (Global stability)**.** *Assume that $f(t, Jy(t))$ is a uniformly bounded function, so that $|f(t, Jy(t)| < M$ for every t with $M > 0$. Then the stochastic system* (3.29) *is globally stable in second moment whenever $\gamma < \dfrac{-(K^2\tau + 1)}{2}$.*

Proof. Firstly define $\bar{y}(t) = y(t) - \tilde{y}(t)$. Therefore,

$$\begin{aligned} D_+\left(\mathbb{E}|\bar{y}(t)|^2\right) &= \lim_{h\to 0} \frac{\mathbb{E}|\bar{y}(t+h)|^2 - \mathbb{E}|\bar{y}(t)|^2)}{h} \\ &= \lim_{h\to 0} \frac{\mathbb{E}(|\bar{y}(t+h)|^2 - |\bar{y}(t)|^2)}{h} \\ &\leq \mathbb{E}\left(\lim_{h\to 0}\left|\frac{|\bar{y}(t+h)|^2 - |\bar{y}(t)|^2}{h}\right|\right) \\ &\leq \mathbb{E}\left(\lim_{h\to 0}\left|\frac{D_+(\bar{y}(t)^2)h + o(h)}{h}\right|\right) \\ &\leq \mathbb{E}\left(D_+(\bar{y}(t)^2)\right). \end{aligned}$$

The last term exists and is well defined from (3.29) and the uniform boundedness of f. Therefore $D_+\left(\mathbb{E}\left(\bar{y}(t)\right)^2\right)$ exists, and we can apply usual calculus rules. Proceeding like this,

$$\begin{aligned} D_+\left(\bar{y}(t)\right)^2 &= 2\bar{y}(t)D_+\left(\bar{y}(t)\right) = 2\bar{y}(t)D_+(y(t) - \tilde{y}(t)) \\ &= 2\bar{y}(t)D_+(y(t) - \tilde{y}(t)) \\ &= 2\bar{y}(t)D_+y(t) - 2\bar{y}(t)D_+\tilde{y}(t) \\ &= 2\bar{y}(t)\left(\gamma y(t) + f(t, Jy(t))\right) - 2\tilde{y}(t)\left(\gamma\tilde{y}(t) + f(t, J\tilde{y}(t))\right) \\ &= 2\gamma\bar{y}(t)^2 + 2\bar{y}(t)\left(f(t, Jy(t)) - f(t, J\tilde{y}(t))\right). \end{aligned}$$

Taking expectation on both sides,

$$D_+\mathbb{E}\left(\bar{y}(t)\right)^2 \leq 2\gamma\mathbb{E}\bar{y}(t)^2 + \mathbb{E}y(t)^2 + \mathbb{E}(f(t, Jy(t)) - f(t, J\tilde{y}(t)))^2. \tag{3.34}$$

Now applying the linear growth condition of the function f,

$$D_+\mathbb{E}\left(\bar{y}(t)\right)^2 \leq 2\gamma\bar{y}(t)^2 + \mathbb{E}(y(t)^2) + K^2\mathbb{E}\left(\int\limits_{t-\tau}^{t}(y(t)-\tilde{y}(t))\mathrm{d}B(s)\right)^2$$
$$\leq (2\gamma+1)\mathbb{E}\bar{y}(t)^2 + K^2\tau \sup_{t-\tau\leq s\leq t}\mathbb{E}(\bar{y}(t))^2.$$

Take $A = -(2\gamma+1)$ and $B = K^2\tau$. For applying Halanay's theorem, we need

1. $A > 0 \Rightarrow \gamma < \dfrac{-1}{2}$.

2. $-A + B < 0 \Rightarrow (2\gamma+1) + K^2\tau < 0$. Therefore $\gamma < \dfrac{-(K^2\tau+1)}{2}$.

Therefore, for every $\gamma < \dfrac{\min\left(-1, -K^2\tau - 1\right)}{2}$, all the hypotheses of Halanay's theorem will be satisfied. But $\min\left(-1, -K^2\tau-1\right) = -K^2\tau - 1$ as K^2 and τ are nonnegative. Hence, for every $\gamma < \dfrac{-(K^2\tau+1)}{2}$, we have the following result.

$$\mathbb{E}(y(t)-\tilde{y}(t))^2 \leq \sup_{0\leq s\leq\tau}\mathbb{E}(\varphi(s)-\psi(s))^2. \tag{3.35}$$

which completes the theorem. □

From the above theorem it is clear that the stochastic system defined in (3.29) is globally stable in second moment with $C = 1$ in accordance with the definition 3.5.1. Therefore (3.29) is a *contractive* or *dissipative system* for γ, K and τ satisfying the relation specified by theorem 3.5.3.

Chapter 4

Numerical Analysis

Introduction

In chapter 3, we derived important theoretical properties of general stochastic delay integro differential equations (see section 3.3) like the existence and uniqueness of solutions, square integrability, $\mathbb{L}^p$ boundedness and stability of a subclass of equations. Though the proof of the existence and uniqueness theorem (Theorem 3.3.1) gives a numerical scheme to obtain approximate solutions, for large iterations it is very difficult to continue with this scheme. There are many real life applications in mathematical finance [15], physics [21] and population dynamics [12] which make use of SDIDEs. Because of the inherent complexity of these equations, finding an exact solution is very difficult. Therefore we have to device numerical methods to obtain approximate solutions.

In this chapter, we try to develop an analogous version of the classical Euler - Maruyama method for SDIDE. We divide this chapter into three sections. In the first section, we explain the numerical scheme. A detailed error analysis is provided in the second section, and we obtained that for a bounded interval the error is of the order of the mesh size, i.e., $\delta(y(t)) = \mathcal{O}(h)$, where $\delta(y(t))$ is the error in $y(t)$. In the third section, we illustrates the theoretical results using numerical examples.

4.1 Euler - Maruyama method for SDIDE

We consider a sub class of SDIDE of the following form.

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} & = f(t, y(t)) + G(t, Jy(t)), \quad \forall t \in (\tau, T], \\ y(t) & = \varphi(t), \quad \forall t \in [0, \tau], \\ Jy(t) & = \displaystyle\int\limits_{t-\tau}^{t} g(u, y(u))\, \mathrm{d}B(u), \quad \forall t \in [0, \tau]. \end{cases} \tag{4.1}$$

We assume that the functions $f(t, y(t)), G(t, Jy(t))$ and $g(u, y(u))$ are Lipschitz continuous and linearly growing in their respective domains. Then, theorem 3.3.1 guarantees a unique solution for equation (5.3). In the following section we explain a numerical method called Euler - Maruyama scheme to obtain numerical solutions, which approximate the unique solution guaranteed by theorem 3.3.1, for a the subclass of equations given in (5.3).

4.1.1 Numerical scheme

Assume that the domain of interest is a bounded interval in time $[0, T]$, with $\tau \in [0, T)$. Choose a mesh size h such that $N = \dfrac{T}{h}$ and $M = \dfrac{\tau}{h}$ are both positive integers. With the above choice of h, define the partition of $[0, \tau] \cup (\tau, T]$ as $t_n = (n-1)h$ for $n = 1, 2, \cdots, N+1$. Therefore, $t_1 = 0, t_{M+1} = \tau$ and $t_{N+1} = T$. Figure 4.1 shows the discretization of time interval in which the numerical scheme is implemented. We know

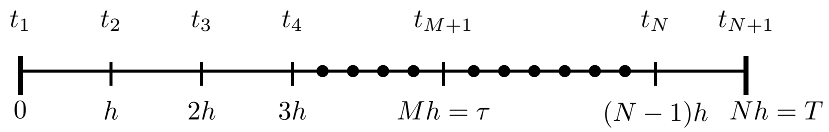

Figure 4.1: General grid on which the numerical schemes are implemented. h denotes the size of the mesh. $N = T/h$ is the number of iterations.

that $y(t) = \varphi(t), \forall t \in [0, \tau]$ and it is given a priori. Therefore the exact solution till the iteration $n = M + 1$ is known to us. Hence by using this information, our aim is to come by the value of $y(t)$ in the next interval $[\tau, \tau + h]$.

Now we introduce a few notations. For any $k \in \{1, 2, \cdots, N+1\}$, y_k denotes the approximate value at t_k and $y(t_k)$ exact value at k. If so, then the Euler - Maruyama scheme is defined as,

$$y_{i+1} = y_i + h(f(t_i, y_i) + G(Jy_i)). \tag{4.2}$$

where $i = M+1, \cdots, N$. In equation (4.2), we can see a term Jy_i appears, which stands the for the approximated delay Itô integral. We define it as,

$$\begin{aligned}
\int_{t_i-\tau}^{t_i} g(s, y(s))\mathrm{d}B(s) &= \sum_{k=i-M}^{i-1} \int_{t_k}^{t_{k+1}} g(s, y(s))\mathrm{d}B(s). \\
&\approxeq \sum_{k=i-M}^{i-1} g(t_k, y_k)(B(t_{k+1}) - B(t_k)). \\
&\approxeq \sum_{k=i-M}^{i-1} g(t_k, y_k)\zeta_k = Jy_i,
\end{aligned}$$

where $\zeta_k \approx N(0, h)$. If we assume $i = M+1$, we must know the values $y_{M+1}, y_M, y_{M-1}, \cdots, y_1$ a priori. But the information up to this level is known from equation (5.3), as $y_k = \varphi(t_k)$ for $k \leq M+1$. Hence, the equation (4.2) makes sense. For analytical purpose, it is easier to define equation (4.2) as a piece wise step function.

We define for $i \geq M+1$

$$\bar{y}(t) = \sum_{k=1}^{i} y_k \mathbb{I}_k, \tag{4.3}$$

$$\tilde{y}(t) = \sum_{k=1}^{i} y(t_k) \mathbb{I}_k, \tag{4.4}$$

where $\mathbb{I}_k = \mathbb{I}_{[t_{k-1}, t_k]}$. Then (4.2) can be reformulated as,

$$\bar{y}(t) = \bar{y}(t_i) + \int_{t_i}^{t} f(s, \bar{y}(t_i))\mathrm{d}s + \int_{t_i}^{t} G(J\bar{y}(t_i))\mathrm{d}s, \quad t \in (t_i, t_{i+1}]. \tag{4.5}$$

It is easy to check that the previous equation is consistent with equation (4.2). More over, for the interval $(t_i, t]$ we can express the exact value as,

$$y(t) = y(t_i) + \int_{t_i}^{t} f(t, y(s))\mathrm{d}s + \int_{t_i}^{t} G(Jy(s))\mathrm{d}s. \tag{4.6}$$

For most of the SDIDEs, we can hardly find the analytical solution. Hence in the case of SDIDE, we are not able to compare approximate values with exact values. That is, mean of the error is not a feasible quantification of error for SDIDEs. The above type of convergence in which exact solution itself is close enough to the numerical solution is called the strong convergence. An alternative is to use the error of means which is called the weak convergence. In weak convergence, we are not assured with path wise closeness, but we have the moments of the numerical solution converges to the respective moments of the actual solution. That is, the exact solution and the numerical solution are having the same probability distribution. Errors in the respective cases are called strong error and weak error. We make the above concepts mathematically rigorous next. Define the weak error as,

$$\delta_w(t_k) = \left|\mathbb{E}(y(t_k)) - \langle y_k \rangle\right|. \tag{4.7}$$

where $\langle y_k \rangle$ is the average of y_k over a large number of simulations. For completeness, we specify the strong error also,

$$\delta(t_k) = \mathbb{E}\left|y(t_k) - y_k\right|. \tag{4.8}$$

In addition, we define $\delta_s(t_k) = \mathbb{E}\left|y(t_k) - y_k\right|^2$ as the mean square error. If $\delta_s(t_k) \to 0$, we say the numerical scheme converge in mean square. It is clear that strong convergence ($\delta(t_k) \to 0$) implies weak convergence ($\delta_w(t_k) \to 0$). Note that the converse may not be true. We close this section with the above definitions and in the next section we prove that the error is bounded and grows at most of the order of the mesh size, h.

4.2 Error analysis

Theorem 4.2.1. *If the functions $f(\cdot,\cdot), G(\cdot)$ and $g(\cdot,\cdot)$ satisfy the Lipschitz condition and linear growth condition, then the Euler - Maruyama scheme defined by* (4.2) *in the bounded interval* $[0,T]$ *with initial data given by* (5.3) *converges strongly to the unique solution and the mean square error is at most of the order of the mesh size, h.*

Proof. We mainly use the methods presented in proving the stability of the solutions (see section 3.5). Define the function,

$$H(t) = y(t) - \bar{y}(t),\ \forall t \in [t_i, t_{i+1}]. \tag{4.9}$$

we can apply (4.6) and (4.5) to obtain,

$$
\begin{aligned}
H(t) &= y(t) - \bar{y}(t) \\
&= y(t_i) - \bar{y}(t_i) + \int_{t_i}^{t} \left(f(s, y(s)) - f(s, \bar{y}(s))\right) \mathrm{d}s + \int_{t_i}^{t} \left(G(Jy(s)) - G(J\bar{y}(s))\right) \mathrm{d}s \\
&= H(t_i) + \int_{t_i}^{t} \left(f(s, y(s)) - f(s, \bar{y}(s))\right) \mathrm{d}s + \int_{t_i}^{t} \left(G(Jy(s)) - G(J\bar{y}(s))\right) \mathrm{d}s.
\end{aligned}
$$

We can express the previous expression as,

$$
DH(t) = \left(f(t, y(t)) - f(t, \bar{y}(t))\right) + \left(G(Jy(t)) - G(J\bar{y}(t))\right). \tag{4.10}
$$

where $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$. Hence

$$
DH^2(t) = 2H(t)DH(t).
$$

Applying equation (4.10),

$$
\begin{aligned}
DH^2(t) &= 2H(t)DH(t), \\
&= 2H(t)\left(\left(f(t, y(t)) - f(t, \bar{y}(t))\right) + \left(G(Jy(t)) - G(J\bar{y}(t))\right)\right), \\
&\leq (H(t))^2 + \left(\left(f(t, y(t)) - f(t, \bar{y}(t))\right) + \left(G(Jy(t)) - G(J\bar{y}(t))\right)\right)^2, \\
&\leq (H(t))^2 + 2\left(f(t, y(t)) - f(t, \bar{y}(t))\right)^2 + 2\left(G(Jy(t)) - G(J\bar{y}(t))\right)^2.
\end{aligned}
$$

Therefore,

$$
H^2(t) \leq H^2(t_i) + 2\int_{t_i}^{t} \left(f(s, y(s)) - f(s, \bar{y}(s))\right)^2 \mathrm{d}s + \int_{t_i}^{t} \left(G(Jy(s)) - G(J\bar{y}(s))\right)^2 \mathrm{d}s.
$$

Taking expectation over the above expression,

$$
\mathbb{E}(H^2(t)) \leq \mathbb{E}(H^2(t_i)) + 2\mathbb{E}\int_{t_i}^{t} \left(f(s, y(s)) - f(s, \bar{y}(s))\right)^2 \mathrm{d}s + 2\mathbb{E}\int_{t_i}^{t} \left(G(Jy(s)) - G(J\bar{y}(s))\right)^2 \mathrm{d}s.
$$

Applying the Lipschitz continuity of f and G,

$$\begin{aligned}\mathbb{E}(H^2(t)) &\leq \mathbb{E}(H^2(t_i)) + 2K^2\mathbb{E}\int_{t_i}^{t} (y(s)-\bar{y}(s))^2\,\mathrm{d}s + 2K^2\mathbb{E}\int_{t_i}^{t} (Jy(s)-J\bar{y}(s))^2\,\mathrm{d}s,\\ &\leq \mathbb{E}(H^2(t_i)) + 2K^2\mathbb{E}\int_{t_i}^{t} H^2(s)\mathrm{d}s + 2K^2\mathbb{E}\int_{t_i}^{t} (Jy(s)-J\bar{y}(s))^2\,\mathrm{d}s. \qquad (4.11)\end{aligned}$$

We have to estimate the last term in the above equation. Consider,

$$\begin{aligned}\mathbb{E}\int_{t_i}^{t} (Jy(s)-J\bar{y}(s))^2\,\mathrm{d}s &\leq \mathbb{E}\int_{t_i}^{t} (Jy(s)-J\tilde{y}(s)+J\tilde{y}(s)-J\bar{y}(s))^2\,\mathrm{d}s\\ &\leq 2(t-t_i)\mathbb{E}\int_{t_i}^{t} (Jy(s)-J\tilde{y}(s))^2\,\mathrm{d}s + 2(t-t_i)\mathbb{E}\int_{t_i}^{t} (J\tilde{y}(s)-J\bar{y}(s))^2\,\mathrm{d}s\\ &\leq 2\sup_{[t_i,t_{i+1}]}\mathbb{E}\left(Jy(s)-J\tilde{y}(s)\right)^2 + 2\sup_{[t_i,t_{i+1}]}\mathbb{E}\left(J\tilde{y}(s)-J\bar{y}(s)\right)^2\\ &= 2(t-t_i)\sup_{[t_i,t_{i+1}]}\mathbb{E}\left(\int_{s-\tau}^{s}(g(u,y(u))-g(u,\tilde{y}(u)))\mathrm{d}B(u)\right)^2\\ &\quad + 2(t-t_i)\sup_{[t_i,t_{i+1}]}\mathbb{E}\left(\int_{s-\tau}^{s}(g(u,\bar{y}(u))-g(u,\tilde{y}(u)))\mathrm{d}B(u)\right)^2\\ &\leq 2(t-t_i)K^2\sup_{[t_i,t_{i+1}]}\int_{s-\tau}^{s}\mathbb{E}(y(u)-\tilde{y}(u))^2\,\mathrm{d}s + 2(t-t_i)K^2\sup_{[t_i,t_{i+1}]}\int_{s-\tau}^{s}\mathbb{E}(\tilde{y}(u)-\bar{y}(u))^2\,\mathrm{d}s\\ &\leq 2(t-t_i)K^2\tau\sup_{[t_i,t_{i+1}]}\sup_{[s-\tau,s]}\mathbb{E}(y(u)-\tilde{y}(u))^2 + 2(t-t_i)K^2\tau\sup_{[t_i,t_{i+1}]}\sup_{[s-\tau,s]}\mathbb{E}(\bar{y}(u)-\tilde{y}(u))^2\\ &\leq 2(t-t_i)K^2\tau\sup_{[t_i-\tau,t_{i+1}]}\mathbb{E}(y(u)-\tilde{y}(u))^2 + 2(t-t_i)K^2\tau\sup_{[t_i-\tau,t_{i+1}]}\mathbb{E}(\bar{y}(u)-\tilde{y}(u))^2\\ &\leq 2(t-t_i)K^2\tau\sup_{[t_i-\tau,t_{i+1}]}\mathbb{E}(y(u)-\tilde{y}(u))^2 + 2(t-t_i)K^2\tau\max_{k\leq i}\mathbb{E}(y(t_k)-y_k)^2\\ &\leq 2(t-t_i)K^2\tau\sup_{[t_i-\tau,t_{i+1}]}\mathbb{E}(y(u)-\tilde{y}(u))^2 + 2(t-t_i)K^2\tau\max_{k\leq i}\mathbb{E}H^2(t_k). \qquad (4.12)\end{aligned}$$

The last two steps are because of the piecewise definition of $\tilde{y}(s)$ and $\bar{y}(s)$. Consider the first term in equation (4.12). We have,

$$\sup_{[t_i-\tau,t_{i+1}]} \mathbb{E}(y(u)-\tilde{y}(u))^2 = \sup_{\cup_{i-M\le k\le i}[t_k,\,t_{k+1}]} \mathbb{E}(y(u)-\tilde{y}(u))^2.$$

$$\le \max_{i-M\le k\le i} \sup_{[t_k,t_{k+1}]} \mathbb{E}(y(u)-\tilde{y}(u))^2.$$

$$\le \max_{k\le i} \sup_{[t_k,t_{k+1}]} \mathbb{E}(y(u)-y(t_k))^2. \tag{4.13}$$

Therefore, substituting equation (4.13) in equation (4.12), we get,

$$\mathbb{E}\int_{t_i}^{t} (Jy(s)-J\bar{y}(s))^2\,\mathrm{d}s \le 2(t-t_i)K^2\tau \max_{k\le i} \sup_{[t_k,t_{k+1}]} \mathbb{E}(y(u)-y(t_k))^2 + 2(t-t_i)K^2\tau \max_{k\le i} \mathbb{E}H^2(t_k). \tag{4.14}$$

Finally substituting equation (4.14) in equation (4.11) gives,

$$\mathbb{E}(H^2(t)) \le \mathbb{E}(H^2(t_i)) \;+\; 2K^2\mathbb{E}\int_{t_i}^{t} H^2(s)\mathrm{d}s$$

$$+\,4(t-t_i)K^4\tau \max_{k\le i} \sup_{[t_k,t_{k+1}]} \mathbb{E}(y(u)-y(t_k))^2 \;+\; 4(t-t_i)K^4\tau \max_{k\le i} \mathbb{E}H^2(t_k). \tag{4.15}$$

Applying Gronwal's inequality on equation (4.15), we obtain that,

$$\mathbb{E}(H^2(t)) \le \mathbb{E}(H^2(t_i))\exp\left(\alpha_1(t-t_i)\right)$$

$$+\,\alpha_2(t-t_i)\left(\max_{k\le i} \sup_{[t_k,t_{k+1}]} \mathbb{E}(y(u)-y(t_k))^2 \;+\; \max_{k\le i} \mathbb{E}H^2(t_k)\right)\exp\left(\alpha_1(t-t_i)\right), \tag{4.16}$$

where $\alpha_1 = 2K^2$ and $\alpha_2 = 4K^2\tau$. Now it remains to estimate the max-sup term in equation (4.16). We have,

$$\mathbb{E}(y(s)-y(t_k))^2 \le 2(s-t_k)\mathbb{E}\int_{t_k}^{s} |f(u,y(u))|^2\,\mathrm{d}u + 2(s-t_k)\int_{t_k}^{s} |G(Jy(u))|^2\,\mathrm{d}u.$$

$$\le \alpha_1(s-t_k)\mathbb{E}\int_{t_k}^{s}(1+|y(u)|^2)\mathrm{d}u + \alpha_1(s-t_k)\mathbb{E}\int_{t_k}^{s}(1+|Jy(u)|^2)\mathrm{d}u.$$

$$
\begin{aligned}
&\leq 2\alpha_1(s-t_k)^2 + \alpha_1(s-t_k)\int\limits_{t_k}^{s}\mathbb{E}|y(u)|^2\mathrm{d}u + \alpha_1(s-t_k)\int\limits_{t_k}^{s}\mathbb{E}|Jy(u)|^2\mathrm{d}u.\\
&\leq 2\alpha_1(s-t_k)^2 + \alpha_1(s-t_k)\sup_{[t_k,t_{k+1}]}\mathbb{E}|y(u)|^2 + \alpha_1(s-t_k)\int\limits_{t_k}^{s}\mathbb{E}|Jy(u)|^2\mathrm{d}u.\\
&\leq 2\alpha_1(s-t_k)^2 + \alpha_1(s-t_k)\sup_{[t_k,t_{k+1}]}\mathbb{E}|y(u)|^2 + \frac{1}{2}\alpha_2(s-t_k)^2\left(1+\sup_{[0,t_{k+1}]}\mathbb{E}|y(u)|^2\right).\\
&\leq (s-t_k)\left(2\alpha_1(s-t_k) + \alpha_1\sup_{[t_k,t_{k+1}]}\mathbb{E}|y(u)|^2 + \frac{1}{2}\alpha_2(s-t_k)\left(1+\sup_{[0,t_{k+1}]}\mathbb{E}|y(u)|^2\right)\right).\\
&\leq \mathbb{G}_T\times(s-t_k). \qquad (4.17)
\end{aligned}
$$

Substituting equation (4.17) in equation (4.16) we get,

$$
\begin{aligned}
\mathbb{E}(H^2(t)) &\leq \mathbb{E}(H^2(t_i))\exp\left(\alpha_1(t-t_i)\right)\\
&\quad + \alpha_2(t-t_i)\left(\mathbb{G}_T(t_{k+1}-t_k) \;+\; \max_{k\leq i}\mathbb{E}H^2(t_k)\right)\exp\left(\alpha_1(t-t_i)\right). \qquad (4.18)
\end{aligned}
$$

Therefore,

$$
\mathbb{E}(H^2(t_{k+1})) \leq \mathbb{E}(H^2(t_i))\exp\left(\alpha_1 h\right) + \alpha_2 h\left(\mathbb{G}_T h \;+\; \max_{k\leq i}\mathbb{E}H^2(t_k)\right)\exp\left(\alpha_1 h\right). \qquad (4.19)
$$

The above equation can be reformulated as,

$$
\begin{aligned}
\mathbb{E}(H^2(t_{i+1})) &\leq \mathbb{E}(H^2(t_i))\exp\left(\alpha_1 h\right) + \alpha_2\left(\mathbb{G}_T\times h \;+\; \max_{k\leq i}\mathbb{E}H^2(t_k)\right)\exp\left(\alpha_1 h\right)\\
&\leq \mathbb{E}(H^2(t_i))\exp\left(\alpha_1 h\right) + \alpha_2\mathbb{G}_T h^2 + \alpha_2\exp\left(\alpha_1 h\right)h\max_{k\leq i}\mathbb{E}H^2(t_k),\\
&= \beta_1\mathbb{E}(H^2(t_i)) + \beta_2 h^2 + \beta_3 h\max_{k\leq i}\mathbb{E}H^2(t_k). \qquad (4.20)
\end{aligned}
$$

where $\beta_1 = \exp\left(\alpha_1 h\right), \beta_2 = \alpha_2\mathbb{G}_T$ and $\beta_3 = \alpha_2\exp\left(\alpha_1 h\right)$. Now we follow an inductive argument. We have $\max_{k\leq i}\mathbb{E}H^2(t_k) = \mathbb{E}H^2(t_i)$, since error is increasing function. Then by strong induction,

$$
\begin{aligned}
\mathbb{E}(H^2(t_{i+1})) &\le (\beta_1 + h\beta_3)\mathbb{E}(H^2(t_i)) + \beta_2 h^2 \\
&\le (\beta_1 + h\beta_3)\left((\beta_1 + h\beta_3)\mathbb{E}(H^2(t_{i-1})) + \beta_2 h^2\right) + \beta_2 h^2. \\
&\le (\beta_1 + h\beta_3)^2\mathbb{E}(H^2(t_{i-1})) + \beta_2 h^2(\beta_1 + h\beta_3) + \beta_2 h^2
\end{aligned}
$$

After $k = i - M - 1$ we reach $M + 1$ step, where error is zero. Thus,

$$
\begin{aligned}
\mathbb{E}(H^2(t_{i+1})) &\le \sum_{k=0}^{i-M} \beta_2 h^2(\beta_1 + h\beta_3)^k \\
&\le \beta_2 h^2 \frac{(\beta_1 + h\beta_3)^{i-M+1} - 1}{\beta_1 + h\beta_3 - 1} \\
&\le \frac{\beta_2 h^2}{\beta_1 + h\beta_3 - 1}\left((\beta_1 + h\beta_3)^{i-M+1} - 1\right).
\end{aligned}
$$

Since $\beta_1 - 1 > 1$, we have,

$$
\begin{aligned}
\mathbb{E}(H^2(t_{i+1})) &\le \frac{h\beta_2}{\beta_3}(\exp\left((\beta_1 + h\beta_3 - 1)(i - M + 1)\right) - 1) \\
&\le \frac{h\beta_2}{\beta_3}(\exp\left((\beta_1 + h\beta_3 - 1)(N - M + 1)\right) - 1).
\end{aligned}
$$

Note that $\beta_1 - 1 = \exp(\alpha_1 h) - 1$. We can always choose h sufficiently small such that $\beta_1 - 1 < h$. Therefore, for this choice of h,

$$
\begin{aligned}
\mathbb{E}(H^2(t_{i+1})) &\le \frac{h\beta_2}{\beta_3}(\exp\left((1 + \beta_3)h(N - M + 1)\right) - 1). \\
&\le \frac{h\beta_2}{\beta_3}(\exp\left((1 + \beta_3)Nh\right) - 1). \\
&\le \frac{h\beta_2}{\beta_3}(\exp\left((1 + \beta_3)T\right) - 1) \;\; \forall i < N + 1. \qquad (4.21)
\end{aligned}
$$

Equation (4.21) gives a global bound for the error $\mathbb{E}(y(t_k) - y_k)^2$. Therefore the theorem is proved. $\square$

4.3 Numerical Illustrations

In this section we illustrate the results obtained in the previous sections using two examples. All the calculations were carried out in Matlab R2011b, and important codes are given in a separate appendix. For all the four test cases we considered 5 different mesh sizes $h = 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}$ and 2^{-5}. For each h, we simulated 3000 paths and the average of numerical solution at the time instant $t = T$ is found. Then, the above obtained value is compared with the exact expected value, which can be analytically computed. After completing this step, we plot the weak error defined as $\delta_w(y_N) = ||\mathbb{E}(y(T))| - |\langle y_N \rangle||$ ($< y_N >$ is the computed average) against the mesh size. Since the deviation under our concern in weak error, we expect the order of convergence to be lower than that of the order of convergence of mean square error. Our findings corroborate this hypothesis since the oder of convergence for examples are 0.032 and 0.027 respectively, which is much lower than 1 (Theorem 4.2.1). Hence the term weak convergence is justified.

4.3.1 Example 1

We consider the equation,

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} & = 1 + \displaystyle\int\limits_{t-1}^{t} y(s)\mathrm{d}B(s), \quad \forall 1 < t \leq 2, \\ y(t) & = 1 + t, \quad 0 \leq t \leq 1. \end{cases} \tag{4.22}$$

Taking expectation on both sides of equation (4.22), we obtain the following equation,

$$\frac{\mathrm{d}\mathbb{E}(y(t))}{\mathrm{d}t} = 1. \tag{4.23}$$

Solving it, we obtain the expected value at time t as, $\mathbb{E}(y(t)) = 1 + t$. Therefore, for time $t = 2$, $\mathbb{E}(y(2)) = 3$. The numerical results are shown in Table 4.1.

Figure 4.2 shows the variation of the error with respect to mesh size. We can clearly see that for low mesh size, error is also low. To find the weak order of convergence, we plotted logarithm (to the base 2) of the error against logarithm (to the base 2) of the mesh size in figure 4.3. We took the best linear fit for both the curves. Slope of the best linear fit for logarithmic variation gives the weak order of convergence. From the best linear fit for logarithmic variation, we get the order of convergence as 0.0320. Equations of linear fits, norm of residuals (Δ) and figures are given below.

Mesh size (h)	Calculated expectation ($\langle y_N \rangle$)	Weak error ($\delta_w(t_{N+1})$)
2^{-9}	3.0085	8.5528e − 3
2^{-8}	3.0087	8.7106e − 3
2^{-7}	3.0088	8.8121e − 3
2^{-6}	3.0090	9.0102e − 3
2^{-5}	3.0094	9.3978e − 3

Table 4.1: Expected value of the numerical solution for equation (4.22) at time $t = 2$ is given in the second column. Deviation of the numerical value from the exact value is given in the third column.

$$\delta_w(h) = 0.0270h + 0.0085, \;\; \Delta = 8.7647 \times 10^{-4}.$$

$$\log \delta_w(h) = 0.0320 \log(h) - 6.5888, \;\; \Delta = 0.0255.$$

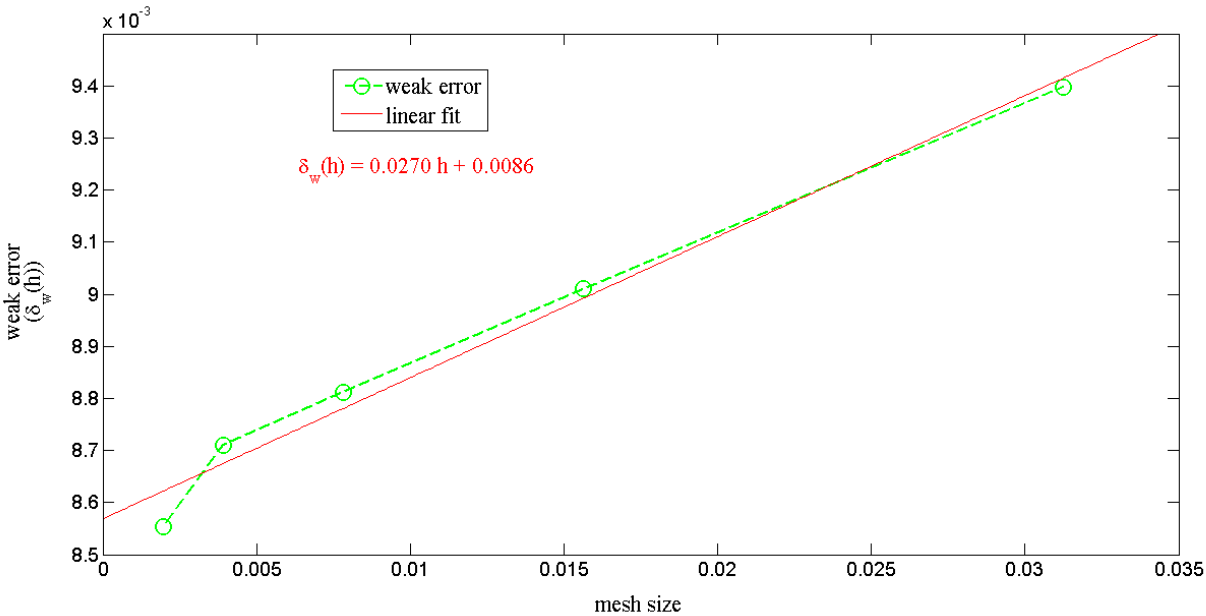

Figure 4.2: Plot showing the variation of weak error with respect to mesh size for test equation (4.22).

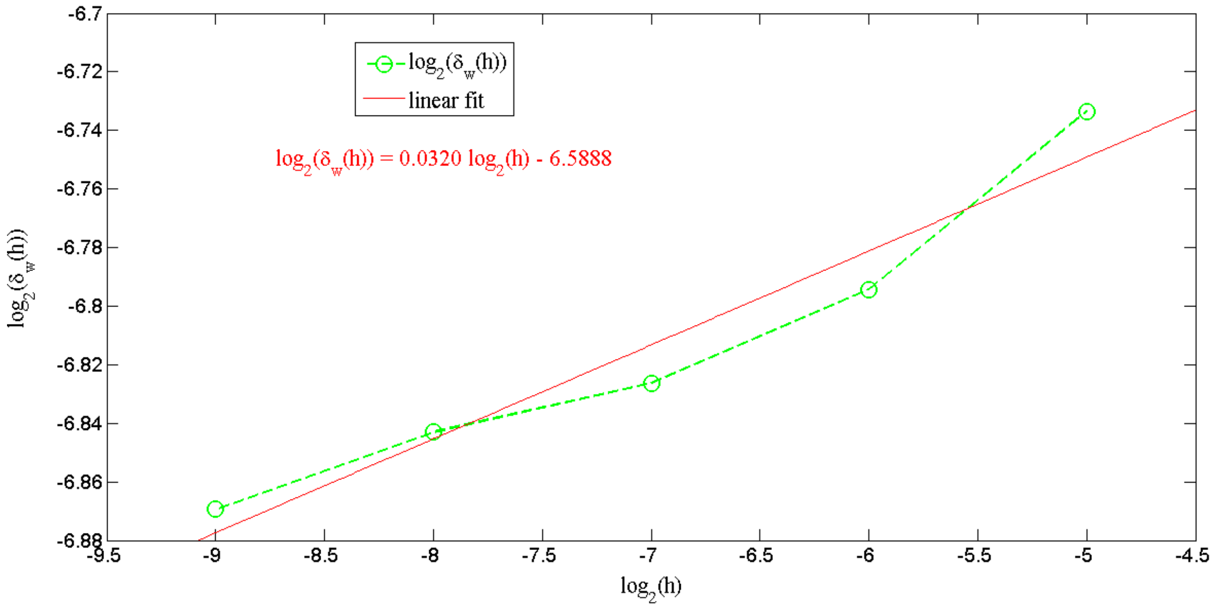

Figure 4.3: Plot showing the variation of logarithm of the weak error with respect to logarithm of the mesh size for test equation (4.22).

4.3.2 Example 2

We consider the equation,

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} &= t + \displaystyle\int_{t-1}^{t} y(s)\mathrm{d}B(s), \quad \forall 1 < t \leq 2, \\ y(t) &= 1 + t, \quad 0 \leq t \leq 1. \end{cases} \tag{4.24}$$

Taking expectation on both sides of equation (4.24), we obtain the following equation,

$$\frac{\mathrm{d}\mathbb{E}(y)}{\mathrm{d}t} = t. \tag{4.25}$$

Solving it, we obtain the expected value at time t as, $\mathbb{E}(y(t)) = \dfrac{t^2+3}{2}$. Therefore, for time $t = 2$, $\mathbb{E}(y(2)) = \dfrac{7}{2}$. The numerical results are shown in Table 4.2. In example (4.24) also low mesh size gives low error. We repeated the calculations done for example (4.22). The weak order of convergence is obtained to be 0.0273. Equations of linear fits, norm of residuals (Δ) and figures are given below.

Mesh size (h)	Calculated expectation ($\langle y_N \rangle$)	Weak error ($\delta_w(t_{N+1})$)
2^{-9}	3.0089	8.9152e − 3
2^{-8}	3.5089	9.0700e − 3
2^{-7}	3.5090	9.1706e − 3
2^{-6}	3.5092	9.3470e − 3
2^{-5}	3.5096	9.6560e − 3

Table 4.2: Expected value of the numerical solution for equation (4.24) at time $t = 2$ is given in the second column. Deviation of the numerical value from the exact value is given in the third column.

$$\delta_w(h) = 0.0234h + 0.0089, \ \Delta = 1.0164 \times 10^{-4}.$$

$$\log \delta_w(h) = 0.0274 \log(h) - 6.5681, \ \Delta = 0.0174.$$

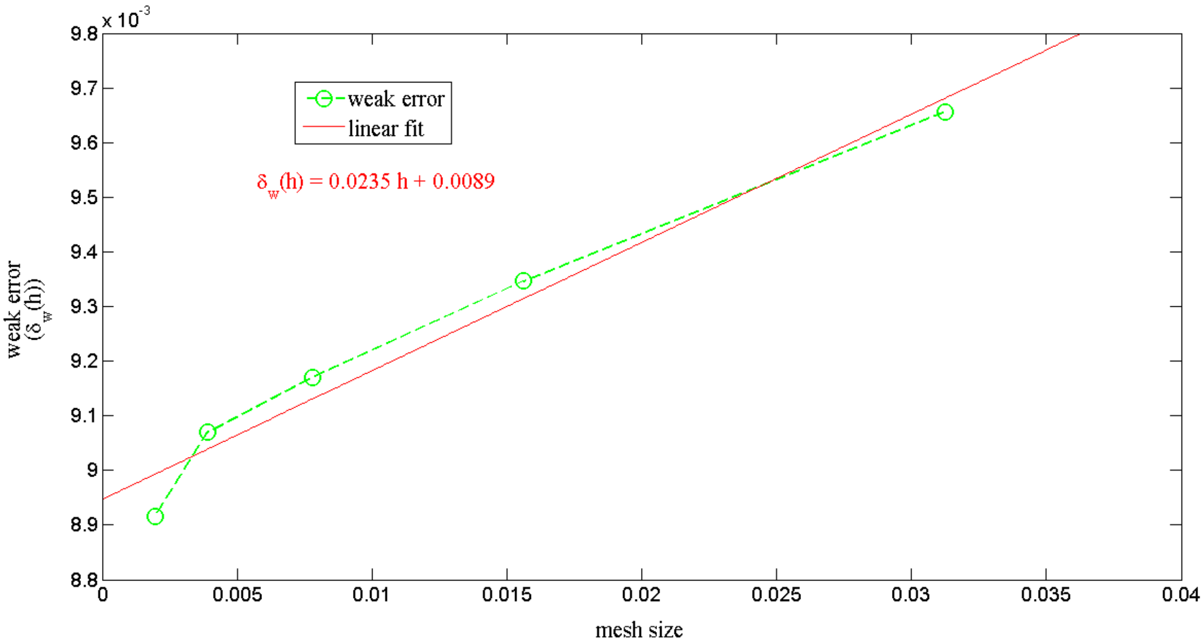

Figure 4.4: Plot showing the variation of weak error with respect to mesh size for test equation (4.24).

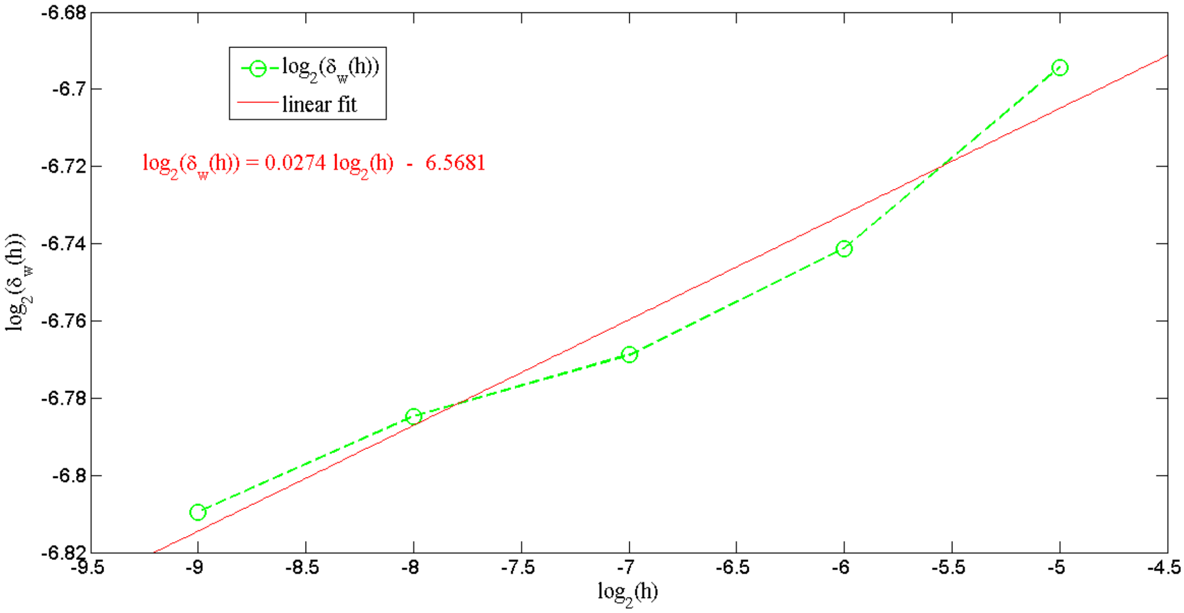

Figure 4.5: Plot showing the variation of logarithm of the weak error with respect to logarithm of the mesh size for test equation (4.24).

Conclusion

In the above two examples, viz. equation (4.22) and (4.24), we have compared the first moment of the numerical solution and the exact solution. We expect it to be lower than the strong order of convergence, which is found to be of the order of the mesh size by Theorem 4.2.1. Consistent with it, we obtained the weak order of convergence to be around 0.03.

Chapter 5

Volatility model

5.1 Introduction

We studied theoretical and numerical features of stochastic delay integro differential equations (SDIDE) in chapter 3 and chapter 4 respectively. The purpose of the present chapter is to study the volatility process as a stochastic delay integro differential equation. We apply the tools developed in chapter 3 and chapter 4 to solve the SDIDE given below. This equation was first put forward by Yuriy Kazmerchuk to study the variation of volatility in financial market [15].

$$\frac{\mathrm{d}\sigma^2(t,S_t)}{\mathrm{d}t} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s,S_s)\,\mathrm{d}B(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t). \tag{5.1}$$

In the above equation α, γ and V are positive constants. τ is the delay in time and $S_t = \{S(t+\theta) : -\tau \le \theta \le 0\}$. $S(t)$ is price of the underlying asset at time t. Substitute $\sigma^2(t,S_t) = y(t)$ in equation (5.1). Then equation (5.1) is changed to the following form.

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sqrt{y(s)}\,\mathrm{d}B(s)\right]^2 - (\alpha+\gamma)y(t). \tag{5.2}$$

We recollect the general form of stochastic delay integro differential equation. It was well explained in section 3.3 of chapter 3.

$$
\begin{aligned}
\mathrm{d}y(t) \;=\; & F\left(t, y(t), \int\limits_{t-\tau}^{t} f(s,y(s))\,\mathrm{d}s, \int\limits_{t-\tau}^{t} g(s,y(s))\,\mathrm{d}B(s)\right)\,\mathrm{d}t \\
& + G\left(t, y(t), \int\limits_{t-\tau}^{t} f(s,y(s))\,\mathrm{d}s, \int\limits_{t-\tau}^{t} g(s,y(s))\,\mathrm{d}B(s)\right)\,\mathrm{d}B(t).
\end{aligned}
\tag{5.3}
$$

We compare equation (5.3) and equation (5.2). Then we understand that equation (5.2) is a particular case of equation (5.3) where,

$$
\begin{aligned}
F\left(t, y(t), \int\limits_{t-\tau}^{t} f(s,y(s))\,\mathrm{d}s, \int\limits_{t-\tau}^{t} g(s,y(s))\,\mathrm{d}B(s)\right) &= \\
\gamma V \;+\; \frac{\alpha}{\tau}\left[\int\limits_{t-\tau}^{t} g(s,y(s))\,\mathrm{d}B(s)\right]^2 &- (\alpha+\gamma)y(t), \\
G\left(t, y(t), \int\limits_{t-\tau}^{t} f(s,y(s))\,\mathrm{d}s, \int\limits_{t-\tau}^{t} g(s,y(s))\,\mathrm{d}B(s)\right) &= 0, \\
f(s,y(s)) &= 0, \\
g(s,y(s)) &= \sqrt{y(s)}.
\end{aligned}
$$

Firstly, we have to show the existence of the solution for equation (5.2). This is done in a step wise fashion and the steps are explained below.

Step 1. Observe that in equation (5.2), $g(s,y(s)) = \sqrt{y(s)}$. If $y(s) < 0$, then $g(s,y(s))$ becomes a complex number. In that case equation (5.2) does not make any sense. Therefore, first of all we must show that $y(s)$ is non negative.

Step 2. In the second step we prove that a solution of equation (5.2) will be bounded and continuous.

Step 3. Using step 2, we show that the stochastic delay integral in equation (5.2) has a unique solution with help of Theorem 3.3.1.

In order to establish step 1, we state and prove a proposition. The proof for this proposition is adapted from X. Mao (section 9.2, chapter 9) [18].

Proposition 5.1.1. *If $y(s) \geq 0$ for every $s \in [0, \tau]$, then any solution of the equation (5.2) with $\gamma V > 1$ is non negative for every $t \in [\tau, T]$ almost everywhere.*

Proof. We have to show that for every t, $y(t) \geq 0$ almost everywhere, provided $y(s) \geq 0$ for every $s \in [0, \tau]$. Let $a_0 = 1$ and $a_k = \exp(-k(k+1))/2$, for every $k \geq 1$. Then,

$$\begin{aligned}\int_{a_k}^{a_{k-1}} \frac{\mathrm{d}u}{u} &= \log\left(\frac{a_{k-1}}{a_k}\right). \\ &= \log\left(\frac{\exp(-(k-1)k/2)}{\exp(-k(k+1)/2)}\right). \\ &= \log(\exp k) = k.\end{aligned}$$

Let $\psi_k(u)$ be a continuous function such that,

1. $supp(\psi_k) \subset (a_k, a_{k-1})$.
2. $0 \leq \psi_k(u) \leq \dfrac{2}{ku}$.
3. $\displaystyle\int\limits_{a_k}^{a_{k-1}} \psi_k(u)\,\mathrm{d}u = 1$.

Define the function $\phi_k(x)$ as,

$$\phi_k(x) = \begin{cases} 0 \ \ \forall x \geq 0. \\ \displaystyle\int_0^{-x} \mathrm{d}y \int_0^{y} \psi_k(u)\mathrm{d}u \ \ \forall x < 0. \end{cases}$$

By definition $\phi_k \in \mathbb{C}^2(\mathbb{R}, \mathbb{R})$, and the first and second derivatives are given by,

$$\begin{aligned}\phi_k'(x) &= -\int\limits_0^{-x} \psi_k(u)\mathrm{d}u. \\ \phi_k''(x) &= \psi_k(-x).\end{aligned}$$

Define $\bar{x} = -x$ if $x < 0$. If $x \geq 0$, define $\bar{x} = 0$. We claim that for every $x \in \mathbb{R}$, $\bar{x} - a_{k-1} \leq \phi_k(x) \leq \bar{x}$. If $x \geq 0$, then $\phi_k(x) = 0$. In this case the claim holds correct. Since $supp(\psi_k) \subset (a_k, a_{k-1})$, for every $x > -a_k$, $\phi_k'(x) = 0$. Again, by condition 3, $-1 \leq \phi_k'(x)$. Therefore we obtain,

$$-1 \leq \phi_k'(x) \leq 0 \ \ \forall -\infty < x < -a_k. \tag{5.4}$$

Hence,

$$\bar{x} - a_{k-1} = \int_{-x}^{a_{k-1}} -1\mathrm{d}u \leq \int_{-x}^{0} -1\mathrm{d}u = \int_{0}^{x} -1\mathrm{d}u \leq \int_{0}^{x} \phi_k'(x) = \phi_k(x). \tag{5.5}$$

Assume that $y(s) < 0$ for $s \in (\tau, T]$. Then, we have,

$$\begin{aligned}
\mathrm{d}\phi_k(y(t)) &= \phi_k'(y(t))y'(t) \\
&= \phi_k'(y(t)) \times \left(\gamma V + \frac{\alpha}{\tau}\left[\int\limits_{t-\tau}^{t} \sqrt{y(s)}\,\mathrm{d}B(s)\right]^2 - (\alpha+\gamma)y(t)\right)\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi_k(y(t)) &= \phi_k(y(\tau)) + \int_{\tau}^{t} \phi_k'(y(s)) \times \left(\gamma V + \frac{\alpha}{\tau}\left[\int\limits_{s-\tau}^{s} \sqrt{y(u)}\,\mathrm{d}B(u)\right]^2 - (\alpha+\gamma)y(s)\right)\mathrm{d}s \\
&= \gamma V \int_{\tau}^{t} \phi_k'(y(s))\mathrm{d}s - (\alpha+\gamma)\int_{\tau}^{t} \phi_k'(y(s))y(s)\mathrm{d}s \\
&\qquad\qquad + \frac{\alpha}{\tau}\int_{\tau}^{t}\left[\int\limits_{s-\tau}^{s} \sqrt{y(u)}\,\mathrm{d}B(u)\right]^2 \phi_k'(y(s))\mathrm{d}s. \\
&\leq \gamma V \phi_k(y(t)) - (\alpha+\gamma)\int_{\tau}^{t} \bar{y}(s)\mathrm{d}(s) \\
&\qquad\qquad - \frac{\alpha}{\tau}\int_{\tau}^{t}\left[\int\limits_{\tau}^{s} \sqrt{y(u)}\,\mathrm{d}B(u)\right]^2 \mathrm{d}s.
\end{aligned}$$

Therefore,

$$\phi_k(y(t)) \leq \frac{1}{1-\gamma V}\left(-(\alpha+\gamma)\int_{\tau}^{t} \bar{y}(s)\mathrm{d}(s) - \frac{\alpha}{\tau}\int_{\tau}^{t}\left[\int\limits_{\tau}^{s} \sqrt{y(u)}\,\mathrm{d}B(u)\right]^2 \mathrm{d}s\right)$$

Hence,

$$\bar{y}(t) - a_{k-1} \leq \phi_k(y(t)) \leq \frac{1}{1-\gamma V}\left(-(\alpha+\gamma)\int_{\tau}^{t} \bar{y}(s)\mathrm{d}(s) - \frac{\alpha}{\tau}\int_{\tau}^{t}\left[\int\limits_{\tau}^{s} \sqrt{y(u)}\,\mathrm{d}B(u)\right]^2 \mathrm{d}s\right).$$

$$\mathbb{E}(\bar{y}(t)) - a_{k-1} \leq \mathbb{E}(\phi_k(y(t))) \leq \frac{\alpha+\gamma}{\gamma V - 1}\int_{\tau}^{t} \mathbb{E}\bar{y}(s)\mathrm{d}s - \frac{\alpha}{\tau(1-\gamma V)}\int\limits_{\tau}^{t}\int\limits_{\tau}^{s} \mathbb{E}|y(u)|\,\mathrm{d}u\,\mathrm{d}s.$$

Since $|y(u)|$ is a positive function, and $1-\gamma V<0$, we obtain,

$$\mathbb{E}(\bar{y}(t)) \leq a_{k-1} + \frac{\alpha+\gamma}{\gamma V-1}\int_\tau^t \mathbb{E}\bar{y}(s)\mathrm{d}s - \frac{\alpha(t-\tau)}{\tau(1-\gamma V)}\int_\tau^t \mathbb{E}|y(s)|\,\mathrm{d}s. \tag{5.6}$$

Therefore,

$$\mathbb{E}(\bar{y}(t)) \leq a_{k-1} + g\int_\tau^t \mathbb{E}\bar{y}(s)\mathrm{d}(s) \tag{5.7}$$

where $g = \dfrac{\gamma\tau+\alpha t}{\tau(\gamma V-1)} > 0$. Applying Gronwall's inequality, we obtain,

$$\mathbb{E}(\bar{y}(t)) \leq a_{k-1}\exp\left(g(t-\tau)\right) \tag{5.8}$$

Taking the limit $k\to\infty$, we get $\mathbb{E}(\bar{y}(t)) \leq 0 \Rightarrow \mathbb{E}(\bar{y}(t)) = 0$. Therefore $\mathbb{P}(y(t)<0)=0$ for all $t>\tau$. Hence $y(t)$ is non negative almost everywhere. This completes the theorem. □

We have achieved the objective of step 1 by proposition 5.1.1. In the next step we prove that any solution of equation(5.2) is bounded. We state and prove it as a separate lemma.

Lemma 5.1.2. *If $y(t)$ is a solution of equation (5.2) in the interval $[\tau,T]$, then*

$$\mathbb{E}\left(\sup_{\tau\leq t\leq T}|y(s)|^2\right) \leq C(T), \tag{5.9}$$

where $C(T)$ is a constant that depends on T.

Proof. To make calculations easier we can re write equation (5.2) as,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a + by(t) + c\left[\int_{t-d}^{t}\sqrt{y(s)}\mathrm{d}B(s)\right]^2. \tag{5.10}$$

Let $y(t)$ be a solution of equation (5.10). Define, $y_n(t) = y(T\wedge\chi_n)$, where $\chi_n = T\wedge \inf\{t\in[\tau,T]:|y_n(t)|\geq n\}$. Then as $n\to\infty$, $\chi_n\to T$. In addition, $y_n(t)$ satisfies the following integral equation.

$$y_n(t) = y_n(d) + a(t-d) + b\int_d^t y_n(s)\mathrm{d}s + c\int_d^t\left[\int_{s-d}^s\sqrt{y_n(u)}\mathrm{d}B(u)\right]^2\mathrm{d}s.$$

Therefore,

$$|y_n(t)|^2 \leq 4|y_n(d)|^2 + 4a^2(t-d)^2 + 4b^2(t-d)\int_d^t |y_n(s)|^2\,\mathrm{d}s$$
$$+ 4c^2(t-d)\int_d^t \left[\int_{s-d}^s \sqrt{y_n(u)}\mathrm{d}B(u)\right]^4 \mathrm{d}s.$$

Taking supremum over the interval [d,t],

$$\sup_{[d,t]}|y_n(t)|^2 \leq 4|y_n(d)|^2 + 4a^2(t-d)^2 + 4b^2(t-d)\int_d^t |y_n(s)|^2\,\mathrm{d}s$$
$$+ 4c^2(t-d)\sup_{[d,t]}\int_d^s \left[\int_{u-d}^u \sqrt{y_n(v)}\mathrm{d}B(v)\right]^4 \mathrm{d}u.$$

Taking expectation on both sides,

$$\mathbb{E}\left(\sup_{[d,t]}|y_n(t)|^2\right) \leq 4\mathbb{E}|y_n(d)|^2 + 4a^2(t-d)^2 + 4b^2(t-d)\int_d^t \mathbb{E}\left(\sup_{[d,s]}|y_n(v)|^2\right)\mathrm{d}s$$
$$+ 4c^2(t-d)\,\mathbb{E}\int_d^t \left[\int_{u-d}^u \sqrt{y_n(v)}\mathrm{d}B(v)\right]^4 \mathrm{d}u.$$

Therefore,

$$\begin{aligned}
\mathbb{E}\left(\sup_{[d,t]}|y_n(t)|^2\right) &\leq 4\mathbb{E}|y_n(d)|^2 + 4a^2(t-d)^2 + 4b^2(t-d)\int_d^t \mathbb{E}\left(\sup_{[d,s]}|y_n(v)|^2\right)\mathrm{d}s \\
&\quad + 4c^2(t-d)\int_d^t 36\times d\int_{s-d}^s |y_n(u)|^2\mathrm{d}u\,\mathrm{d}s. \\
&= 4\mathbb{E}|y_n(d)|^2 + 4a^2(t-d)^2 + 4b^2(t-d)\int_d^t \mathbb{E}\left(\sup_{[d,s]}|y_n(v)|^2\right)\mathrm{d}s \\
&\quad + 144c^2d^2(t-d)\int_d^t\int_{s-d}^s \mathbb{E}|y_n(u)|^2\mathrm{d}u\,\mathrm{d}s. \\
&\leq 4\mathbb{E}|y_n(d)|^2 + 4a^2(t-d)^2 + 4b^2(t-d)\int_d^t \mathbb{E}\left(\sup_{[d,s]}|y_n(v)|^2\right)\mathrm{d}s \\
&\quad + 144c^2d^2(t-d)^2\int_0^t \mathbb{E}|y_n(s)|^2\mathrm{d}s. \\
&\leq 4\mathbb{E}|y_n(d)|^2 + 4a^2(t-d)^2 + 4b^2(t-d)\int_d^t \mathbb{E}\left(\sup_{[d,s]}|y_n(v)|^2\right)\mathrm{d}s \\
&\quad 144c^2d^3(t-d)^2\sup_{[0,d]}\mathbb{E}|\phi(s)|^2 + 144c^2d^2(t-d)^2\int_d^t \mathbb{E}|y_n(s)|^2\mathrm{d}s.
\end{aligned}$$

$$\leq 4\mathbb{E}|y_n(d)|^2 + 4a^2(t-d)^2 + 144c^2d^3(t-d)^2 \sup_{[0,d]} \mathbb{E}|\phi(s)|^2$$
$$\left[4b^2(t-d) + 144c^2d^2(t-d)^2 \int_d^t\right] \mathbb{E}\sup_{[d,s]} |y_n(u)|^2 \mathrm{d}s.$$

Applying Gronwall's inequality to the above equation,

$$\mathbb{E}\left(\sup_{[d,T]} |y_n(s)|^2\right) \leq C(T) \times \exp\left(4b^2(T-d)^2 + 144c^2d^2(T-d)^3\right), \tag{5.11}$$

where $C(T) = 4\mathbb{E}|y_n(d)|^2 + 4a^2(T-d)^2 + 144c^2d^3(T-d)^2 \sup_{[0,d]} \mathbb{E}|\phi(s)|^2$. Passing the limit $n \to \infty$, we obtain the desired result. □

Remark 5.1.3. Note that in the proof of lemma 5.1.2, we did not use any Lipschitz continuity or linear growth condition. Therefore, lemma 5.1.2 is different from the analogous proposition in chapter 3 (see proposition 3.3.2). In equation (5.2), the effect of squaring the delay integral is got nullified by the $\sqrt{y(t)}$ term inside the delay integral. We took advantage of this mutually canceling effect of equation (5.2).

In order to complete step 2, we have to prove that $y(t)$ is continuous for every $t \in [\tau, T]$ almost everywhere. We establish this as an application of *Kolmogorov's continuity criterion*. Firstly, we make a definition, and then state *Kolmogorov's continuity criterion*. The reader is requested to refer *Measure, Probability and Mathematical finance* [11] or lecture notes by N. Berestycki [2] for more information on *Kolmogorov's continuity criterion.*

Definition 5.1.4 (Modification.)**.** A stochastic process $X(t,\omega)$ is called a modification of the stochastic process $Y(t)$ if, for every t,

$$X(t,\omega) = Y(t,\omega) \;\; a.s. \tag{5.12}$$

Lemma 5.1.5 (Kolmogorov's continuity criterion)**.** *Suppose that there exists constants* $\alpha, \beta, K > 0$ *such that,*

$$\mathbb{E}|X(t) - X(s)|^{\alpha} \leq K|t-s|^{1+\beta} \;\; a.s. \tag{5.13}$$

for every t. Then there exists a continuous modification of $X(t)$.

Now, we state and prove any solution of equation (5.10) has a continuous modification.

Proposition 5.1.6. *If* $y(t)$ *is a solution of equation* (5.10)*, then there exists a continuous modification for* $y(t)$*.*

Proof. We shall prove that for any $t_1 > t_2$,

$$\mathbb{E}|y(t_1) - y(t_2)|^2 \leq C \times |t_1 - t_2|^2$$

Then by applying lemma 5.1.5 we obtain the desired result. We have,

$$y(t_1) - y(t_2) = a(t_1 - t_2) + b\int_{t_2}^{t_1} y(s)\,\mathrm{d}s + c\int_{t_2}^{t_1}\left(\int_{s-d}^{s}\sqrt{y(s)}\mathrm{d}B(u)\right)^2 \mathrm{d}s.$$

Therefore,

$$\begin{aligned}
|y(t_1) - y(t_2)|^2 &\leq 3a^2(t_1 - t_2)^2 + 3b^2\left|\int_{t_2}^{t_1} y(s)\,\mathrm{d}s\right|^2 + 3c^2\left|\int_{t_2}^{t_1}\left(\int_{s-d}^{s}\sqrt{y(s)}\mathrm{d}B(u)\right)^2 \mathrm{d}s\right|^2. \\
&\leq 3a^2(t_1 - t_2)^2 + 3b^2(t_1 - t_2)\int_{t_2}^{t_1}|y(s)|^2\,\mathrm{d}s \\
&\qquad + 3c^2(t_1 - t_2)\int_{t_2}^{t_1}\left(\int_{s-d}^{s}\sqrt{y(s)}\mathrm{d}B(u)\right)^4 \mathrm{d}s.
\end{aligned}$$

Taking expectation on both sides we obtain,

$$\mathbb{E}|y(t_1) - y(t_2)|^2 \leq 3a^2(t_1 - t_2)^2 + 3b^2(t_1 - t_2)^2\mathbb{E}\left(\sup_{[0,T]}|y(s)|^2\right) + 108d^2c^2(t_1 - t_2)^2\mathbb{E}\left(\sup_{[0,T]}|y(s)|^2\right).$$

Rearranging the terms we get,

$$\mathbb{E}|y(t_1) - y(t_2)|^2 \leq C \times (t_1 - t_2)^2. \tag{5.14}$$

$C = 3a^2 + \mathbb{E}\left(\sup_{[0,T]}|y(s)|^2\right)(3b^2 + 108d^2c^2)$, which is bounded globally by lemma 5.1.2. □

In the next proposition we assert the existence and uniqueness of equation (5.2).

Proposition 5.1.7. *If equation* (5.2) *satisfies the condition* $\gamma V > 1$*, then it has a unique solution. In addition to that there exists a continuous modification to it.*

Proof. We essentially use ideas in the Theorem 3.3.1. We have,

$$F(t, y(t), Jy(t)) = \gamma V - (\alpha + \gamma)y(t) + (Jy(t))^2 . \tag{5.15}$$

Clearly, $\gamma V - (\alpha + \gamma)y(t)$ is linearly growing and Lipschitz continuous (with respect to $y(t)$). Therefore, it is enough to consider the function,

$$G(t, y(t), Jy(t)) = (Jy(t))^2 . \tag{5.16}$$

Therefore, for the successive approximates to converge, it is enough to get a bound on $\mathbb{E}|y_{n+1}(t) - y_n(t)|^2$ in terms of $|y_{n+1}(t) - y_n(t)|$ with the function (5.16) replacing (5.15). Firstly, we show that the successive approximates are bounded globally. We have,

$$y_n(t) = \gamma V(t - \tau) - (\alpha + \gamma) \int_\tau^t y_{n-1}(s)\, \mathrm{d}s + \frac{\alpha}{\tau} \int_\tau^t \left(\int_{s-\tau}^s \sqrt{y_{n-1}(u)}\, \mathrm{d}B(u) \right)^2 \mathrm{d}s. \tag{5.17}$$

Hence,

$$\begin{aligned}
|y_n(t)|^2 &\le 3\gamma^2 V^2 (t-\tau)^2 + 3(\alpha+\gamma)^2 \left| \int_\tau^t y_{n-1}(s)\, \mathrm{d}s \right|^2 + \\
&\qquad \frac{3\alpha^2}{\tau^2} \left| \int_\tau^t \left(\int_{s-\tau}^s \sqrt{y_{n-1}(u)}\, \mathrm{d}B(u) \right)^2 \mathrm{d}s \right|^2 \\
&\le 3\gamma^2 V^2 (t-\tau)^2 + 3(\alpha+\gamma)^2 (t-\tau) \int_\tau^t |y_{n-1}(s)|^2\, \mathrm{d}s + \\
&\qquad \frac{3\alpha^2}{\tau^2}(t-\tau) \int_\tau^t \left(\int_{s-\tau}^s \sqrt{y_{n-1}(u)}\, \mathrm{d}B(u) \right)^4 \mathrm{d}s \\
\therefore \mathbb{E}|y_n(t)|^2 &\le 3\gamma^2 V^2 (t-\tau)^2 + 3(\alpha+\gamma)^2 (t-\tau) \int_\tau^t \mathbb{E}|y_{n-1}(s)|^2\, \mathrm{d}s \\
&\qquad \frac{3\alpha^2}{\tau^2}(t-\tau) \int_\tau^t \mathbb{E} \left(\int_{s-\tau}^s \sqrt{y_{n-1}(u)}\, \mathrm{d}B(u) \right)^4 \mathrm{d}s \\
&\le 3\gamma^2 V^2 (t-\tau)^2 + 3(\alpha+\gamma)^2 (t-\tau) \int_\tau^t \mathbb{E}|y_{n-1}(s)|^2\, \mathrm{d}s \\
&\qquad \frac{108\alpha^2}{\tau}(t-\tau) \int_\tau^t \int_{s-\tau}^s \mathbb{E}(y_{n-1}(u))^2\, \mathrm{d}u \mathrm{d}s \\
&\le 3\gamma^2 V^2 (t-\tau)^2 + 108\alpha^2 (t-\tau)^2 \sup_{[0,\tau]} \mathbb{E}|\phi(s)|^2 + 3(\alpha+\gamma)^2 (t-\tau) \int_\tau^t \mathbb{E}|y_{n-1}(s)|^2\, \mathrm{d}s \\
&\qquad 108 \frac{\alpha^2}{\tau} (t-\tau)^2 \int_\tau^t \mathbb{E}|y_{n-1}(s)|^2\, \mathrm{d}s.
\end{aligned}$$

Therefore,

$$\mathbb{E}|y_n(t)|^2 \le C_1 + C_2 \int_\tau^t \mathbb{E}|y_{n-1}(s)|^2 \,\mathrm{d}s, \tag{5.18}$$

where,

$$C_1 = 3\gamma^2 V^2 (t-\tau)^2 + 108\alpha^2 (t-\tau)^2 \sup_{[0,\tau]} \mathbb{E}|\phi(s)|^2, \tag{5.19}$$

$$C_2 = 3(\alpha+\gamma)^2 (T-\tau) + 108\frac{\alpha^2}{\tau}(T-\tau)^2. \tag{5.20}$$

Therefore,

$$\begin{aligned}\max_{1\le n\le k} \mathbb{E}|y_n(t)|^2 &\le C_1 + C_2 \int_\tau^t \max_{1\le n\le k} \mathbb{E}|y_{n-1}(s)|^2 \,\mathrm{d}s \\ &\le C_3 + C_2 \int_\tau^t \max_{1\le n\le k} \mathbb{E}|y_{n-1}(s)|^2 \,\mathrm{d}s,\end{aligned}$$

where, $C_3 = C_1 + C_2 \mathbb{E}\sup_{[0,\tau]}|\phi(s)|^2 (T-\tau)$. Therefore, by applying Gronwall's ineqaulity,

$$\max_{1\le n\le k} \mathbb{E}|y_n(t)|^2 \le C_3 \exp\left(C_2(T-\tau)\right). \tag{5.21}$$

Since k is arbitrary,

$$\mathbb{E}|y_n(t)|^2 \le C_3 \exp\left(C_2(T-\tau)\right). \tag{5.22}$$

Therefore the successive approximates are globally bounded. Hence, by applying Hölder's inequality, isometry property, positiveness and boundedness of $y_n(t)$ by proposition 5.1.1[1],

$$\begin{aligned}\mathbb{E}|y_{n+1}(t)-y_n(t)|^2 &\le \mathbb{E}\left|\int_\tau^t \left(\int_{s-\tau}^s \sqrt{y_n(u)}\,\mathrm{d}B(u)\right)^2 \mathrm{d}s - \int_\tau^t \left(\int_{s-\tau}^s \sqrt{y_{n-1}(u)}\,\mathrm{d}B(u)\right)^2 \mathrm{d}s\right|^2 \\ &\le (t-\tau)\mathbb{E}\int_\tau^t \left[\left(\int_{s-\tau}^s \sqrt{y_n(u)}\,\mathrm{d}B(u)\right)^2 - \left(\int_{s-\tau}^s \sqrt{y_{n-1}(u)}\,\mathrm{d}B(u)\right)^2\right]^2 \mathrm{d}s \end{aligned} \tag{5.23}$$

$$\le K^2(t-\tau)\int_\tau^t \mathbb{E}\left|\int_{s-\tau}^s \left(\sqrt{y_n(u)}-\sqrt{y_{n-1}(u)}\right)\,\mathrm{d}B(u)\right|^2 \mathrm{d}s \tag{5.24}$$

$$\le K^2(t-\tau)\int_\tau^t \int_{s-\tau}^s \mathbb{E}\left(\sqrt{y_n(u)}-\sqrt{y_{n-1}(u)}\right)^2 \mathrm{d}u\,\mathrm{d}s$$

$$\le K^2(t-\tau)^2 \int_\tau^t \mathbb{E}|y_n(s)-y_{n-1}(s)|\,\mathrm{d}s.$$

[1]In proposition 5.1.1 we proved non negativeness of y(t) only. But the same arguments in proposition 5.1.1, shows that the successive approximates are also non negative.

Therefore,

$$\mathbb{E}|y_{n+1}(t) - y_n(t)|^2 \leq K^2(t-\tau)^2 \sup_{[\tau,t]} \mathbb{E}|y_n(u) - y_{n-1}(u)|. \tag{5.25}$$

Now, exactly by following the steps of theorem 3.3.1, we obtain the desired conclusion. □

Remark 5.1.8. Note that, to obtain (5.24) from (5.23), we used Lipschitz continuity of the square function with respect to $y_n(t)$. This is perfectly justified since successive approximates are bounded globally for $t \in [0, T]$.

5.2 Numerical illustration

In the previous section we showed that (5.10) has a unique solution. In this section, we use the numerical method developed in chapter 3, namely Euler - Maruyama method (see section 4.1.1) to obtain numerical solutions which approximate the unique true solution to (5.2). Firstly, we recall equation(5.2).

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sqrt{y(s)}\,\mathrm{d}B(s)\right]^2 - (\alpha+\gamma)y(t). \tag{5.26}$$

We discretize equation (5.26) as,

$$\begin{aligned} y_{i+1} &= y_i + \left(\gamma V + \frac{\alpha}{\tau}(Jy_i)^2 - (\alpha+\gamma)y_i\right)h, \\ Jy_i &= \sum_{j=i-1}^{i-M} \sqrt{y_j}\zeta_j. \end{aligned} \tag{5.27}$$

for $i = M+1, M+2, \cdots, N$. Here ζ_j are $N(0,h)$ distributed random variables. Firstly, we present three sample paths generated (figure 5.2) for the equation (5.26) with $\alpha = 0.1, \gamma = 1$ and $V = 2$. We fix delay time as $\frac{1}{2}$ units and mesh size as 0.01. The reader is requested to see section A.2 for details of the program.

If we take expectation on both sides of the equation (5.26), then we obtain the below given equation.

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \gamma V + \frac{\alpha}{\tau}\int_{t-\tau}^{t} u(s)\,\mathrm{d}s - (\alpha+\gamma)u(t). \tag{5.28}$$

where $u(t) = \mathbb{E}y(t)$. Equation (5.28) is a deterministic delay equation. In can be dis-

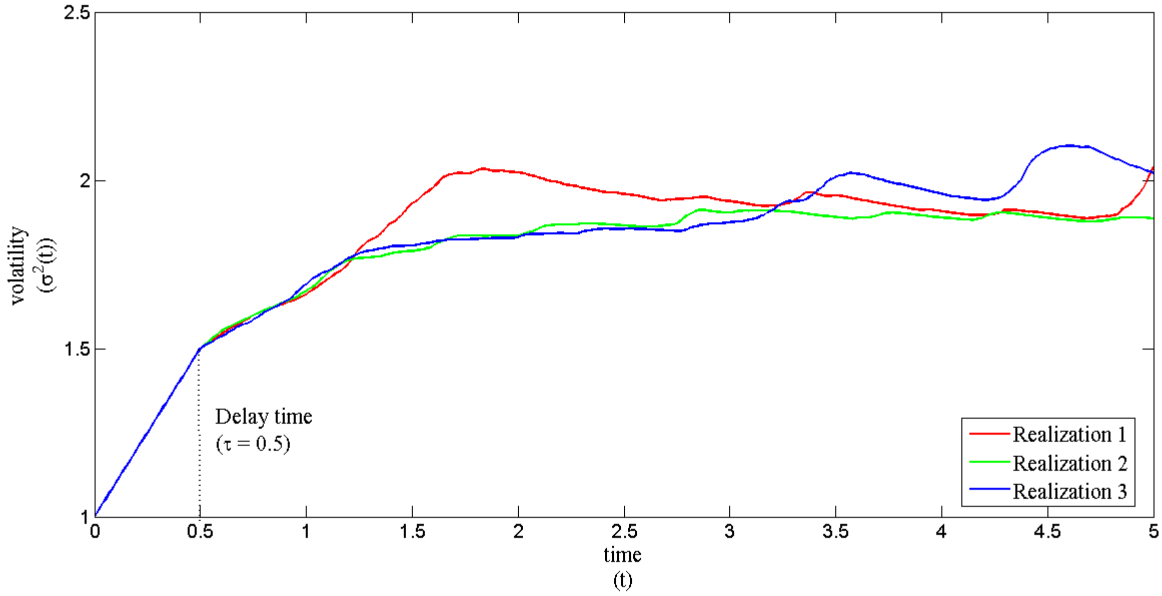

Figure 5.1: Path wise realizations of discretized equation (5.27) with $\alpha = 0.1, \gamma = 1$ and $V = 2$.

cretized as,

$$u_{i+1} = u_i + \left(\gamma V + \frac{\alpha}{\tau} \int_{t_i-\tau}^{t_i} u(s)\,\mathrm{d}s - (\alpha+\gamma)u_i \right) h. \tag{5.29}$$

Here h is the mesh size. With α, γ, τ and V as defined earlier, we solve discretization 5.29. Similarly we sample 5000 paths of discretization 5.29 and calculate the mean value for each t. We compare both, and find they are matching (figure 5.2). The reader is requested to see section A.3 details of the program. Discretization (5.29) and (5.27) are implemented with the mesh size of 0.01. Noe that, the deterministic delay integral in equation (5.29) is calculated using Simpson's $\dfrac{1}{3}$ rule so that, error in evaluating the delay integral will be of the order of 10^{-8} [22]. Figure 5.2 shows the exact matching of the sample average and the expected value calculated using equation (5.29). This indicates the weak convergence of the numerical solution to the exact solution. In the next section, we show an industrial application of equation (5.1).

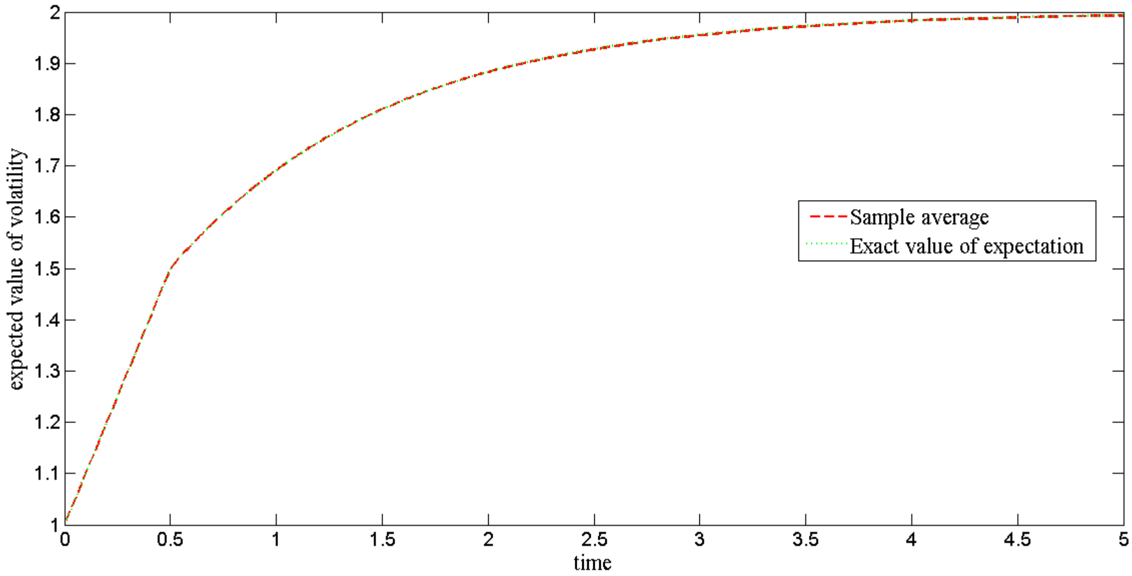

Figure 5.2: Figure showing the sample average of discretization 5.27 and expected value calculated using equation (5.29). Note the exact matching of the two curves.

5.3 Volatility prediction of S&P500 index.

The continuous delay model is of the form,

$$\frac{\mathrm{d}\sigma^2(t,S_t)}{\mathrm{d}t} = \gamma V + \frac{\alpha}{\tau}\left[\int_{t-\tau}^{t} \sigma(s,S_s)\,\mathrm{d}B(s)\right]^2 - (\alpha+\gamma)\sigma^2(t,S_t). \tag{5.30}$$

We consider the variation of stock value of S&P 500 index from January 1, 1992 to December 30, 1993. For this data, equation (5.30) was calibrated by Yuriy Kazmerchuk [15]. We quote the data here,

$$V = 0.007344,\ \gamma = 0.1049,\ \alpha = 0.0446,\ \beta = 0.8505. \tag{5.31}$$

With these information, equation (5.30) is of the form,

$$\frac{\mathrm{d}\sigma^2(t,S_t)}{\mathrm{d}t} = \left(7.7044\times 10^{-4}\right) + \frac{0.0446}{\tau}\left[\int_{t-\tau}^{t} \sigma(s,S_s)\,\mathrm{d}B(s)\right]^2 - 0.1495\sigma^2(t,S_t). \tag{5.32}$$

We predicted the volatility at December 31, 1993, by solving the above equation by the numerical scheme (4.2). We repeat the calculation for different delay times. We found that, for larger delay times, error in the predicted volatility (from actual value) tend to reduce. This substantiates our hypothesis that, considering more historical information will reduce the error in future predictions (see table 5.3). Moreover, we compared the predicted value of the volatility by equation (5.30) with the predicted value by GARCH(1,1) scheme. We define the error as |True value - Predicted value|. GARCH(1,1) scheme is of the form,

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \tag{5.33}$$

We calibrated the coefficients, ω, α and β with the standard program by Prof. John. C. Hull (*http://www-2.rotman.utoronto.ca/ hull/ofod/GarchExample/index.html*). The values obtained are,

$$\omega = 1.3195 \times 10^-6,\ \alpha = 0.050244,\ \beta = 0.910105. \tag{5.34}$$

We found that the error is much lower for prediction by (5.30). This illustrates the robustness of our model. The numerical data is tabulated below. For the GARCH(1,1)

Delay time (days)	Predicted volatility	Absolute error
200	0.6045	0.0134
250	0.5658	0.0254
300	0.5450	0.0460
350	0.5531	0.0379
400	0.5705	0.0205
450	0.6888	0.0977
500	0.6619	0.0708
504	0.6028	0.0117

Table 5.1: Table showing the volatility prediction by equation (5.30) for December 31, 1992. True value of volatility is 0.5911.

process, the predicted volatility is 0.5487. Therefore, the error is 0.0424, which is larger than the predicted value by equation (5.30).

Therefore, from the above calculations we obtained the following results.

- As the delay time increases, error in the predicted volatility tend to reduce, though not monotonically.

- Continuous GARCH model proposed by Yuriy Kazmerchuk gives more accurate value of the future volatility.

The above two results shows the correctness of our hypotheses. Thus, we can see continuous GARCH model is a better tool to predict the future volatility. With this statement, we end chapter 5.

Appendix A

Programs

In this chapter we present important codes and programs developed by us for implementing numerical schemes derived in the previous chapters. All simulations and calculations in this project were carried out in MATLAB R2011b. For easier understanding, all steps are properly commented. Each program is given as a separate section, and a short note is added at the beginning to specify the aims and objectives of the corresponding program.

A.1 Program for error analysis of equation 1

This program samples 3000 paths of the solution of equation 1 for five different step sizes, viz., $h = 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}$ and 2^{-5}. After each run the code will return average value of the solution at the instant $t = 2$ and the error from the exact expected value which can be analytically computed. After the run is initiated, the system will ask the user to input value of a variable p. For a particular value of p, the step size will be 2^{p-10}.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author : Gopikrishnan C. R.                %
% S.visor: Dr. M. P. Rajan                   %
% Subj.  : Numerical solution of SDIDE.      %
% Date   : 4/14/2015                         %
% Venue  : IISER Thiruvananthapuram          %
% Scheme : Comparison of expected value.     %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

randn('state',100)

%%%%%%%%%% Parameters and Variables %%%%%%%%%%

```

```
14 % User can input p, varying from 1 to 5. As p
15 % increases step size increases, by 2 fold.
16
17 prompt = 'Enter the value of p -> ';
18 p = input(prompt);
19
20 h = 2^-9;
21 Tdel  = 1;
22 Tfin  = 2;
23 Tin = 0;
24 N = (Tfin - Tin)/h;
25 M = Tdel/h;
26 R = 2^(p -1);
27 htotal = 0;
28 iter = 3000;
29
30 %%%%%%%%  Euler - Maruyama Iterations %%%%%%%%%
31 for num_iter = 1:iter
32
33 dW = sqrt(h)*randn(1,N);  % White noise is defined as
34                           %   N(0,h) random variables.
35 W = [0,cumsum(dW)];       % Brownian path as a cumulative
36                           %        sum of white noise.
37 H = R*h; K = M/R; L =N/R; % Rescaling delay time and final time.
38 hsoln = ones(1,L+1);      % Solution array.
39 htmesh = [0:H:Tfin];      % Proportional time mesh.
40
41 for i = 1:K+1
42     hsoln(i) = htmesh(i) + 1;    % Definitions of initial condition on [0,Tdel)
43 end
44
45 for i = K+1:L
46     hdelint = 0;
47
48     for j = i - K:i - 1
49         hwinc = sum(dW(R*(j-1) + 1:R*j));       % Rescaled Brownian increment.
50         hdelint = hdelint + hsoln(j)*hwinc;     %  Iterative sum finaly gives
51     end                                         %       the stochastic delay
52                                                 %                   integral.
53     hsoln(i+1) = hsoln(i) + H + hdelint*H;      % E-M iterations.
54 end
55
56 htotal = htotal + hsoln(end);
57 end
58 %%%%%% end of Euler - Maruyama Iterations %%%%%%
59
60 Average_value = htotal/iter       % Return the average value.
61 error = abs(Average_value - 3)    % Return the error.
62
63 %%%%%%%%%%%%%% end of programme %%%%%%%%%%%%%%%%
```

Remark A.1.1. Error analysis of test equation 2 was also carried out using the code A.3. User has to remove the initial condition and discretization of test equation 1 defined in step 44 and 55 respectively and insert the corresponding data for test equation 2. If the delay integral is also different, then the discretization of it defined in step 52 of code A.3 also has to be altered.

A.2 Program for scenario simulation of discretization 5.27

```
(* ::Package:: *)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author : Gopikrishnan C. R.                %
% S.Visor: Dr. M. P. Rajan                   %
% Subj.  : Numerical solution of SDIDE       %
% Date   : 4/14/2015                         %
% Venue  : IISER Thiruvananthapuram          %
% Scheme : Pathwise simulation of            %
%                  volatility equation       %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%% Parameters and Initial Conditions %%%%%%

h = 0.01;
Tdel  = 0.5;
Tfin  = 5;
Tin = 0;
N = (Tfin - Tin)/h;
M = Tdel/h;
R = 1;
gam = 1;
al = 0.1;
V = 2;

%%%%%%%%  Euler - Maruyama Iterations %%%%%%%%%

dW = sqrt(h)*randn(1,N);  % White noise is defined as
                          %   N(0,h) random variables.
W = [0,cumsum(dW)];       % Brownian path as a cumulative
                          %        sum of white noise
H = R*h; K = M/R; L =N/R; % Rescaling delay time and final time.
hsoln = ones(1,L+1);      % solution array
htmesh = [0:H:Tfin];      % proportional time mesh

for i = 1:K+1
    hsoln(i) = htmesh(i) + 1;    % Definitions of initial condition on [0,Tdel)
end

for i = K+1:L
    hdelint = 0;

    for j = i - K:i - 1
        hwinc = sum(dW(R*(j-1) + 1:R*j));            % Rescaled Brownian increment.
        hdelint = hdelint + sqrt(hsoln(j))*hwinc;     %  Iterative sum finaly gives
```

```
    end                                                       %         the stochastic delay
                                                              %                     integral.
    % E-M iterations
    hsoln(i+1) = hsoln(i) + (gam*V + (al/Tdel)*hdelint^2 - (al + gam)*hsoln(i))*H;
end

%%%%%% end of Euler - Maruyama Iterations %%%%%%

%%%%%%%%%%%%%% end of programme %%%%%%%%%%%%%%%%
```

A.3 Program for calculating expected value of discretization 5.27

```
(* ::Package:: *)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Author : Gopikrishnan C. R.                 %
% S.Visor: Dr. M. P. Rajan                    %
% Subj.  : Numerical solution of SDIDE        %
% Date   : 4/14/2015                          %
% Venue  : IISER Thiruvananthapuram           %
% Scheme : Average value of volatility        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%%%%% Parameters and Initial Conditions %%%%%%

h = 0.01;
Tdel  = 0.5;
Tfin  = 5;
Tin = 0;
N = (Tfin - Tin)/h;
M = Tdel/h;
R = 1;
gam = 1;
al = 0.1;
V = 2;
hsolnmat = ones(5000,N+1);    % 5000 by N+1 array to store 3000 samples
                              %       of numerical solution consiting of
                      %                    N+1 discrete values.

%%%%%%%%  Euler - Maruyama Iterations %%%%%%%%%

for iter = 1:5000

dW = sqrt(h)*randn(1,N);   % White noise is defined as
                           %   N(0,h) random variables.
W = [0,cumsum(dW)];        % Brownian path as a cumulative
                           %         sum of white noise
H = R*h; K = M/R; L =N/R;  % Rescaling delay time and final time.
hsoln = ones(1,L+1);       % solution array
htmesh = [0:H:Tfin];       % proportional time mesh

for i = 1:K+1
```

```
    hsoln(i) = htmesh(i) + 1;   % Definitions of initial condition on [0,Tdel)
    hsolnmat(iter,i) = hsoln(i);
end

for i = K+1:L
    hdelint = 0;

    for j = i - K:i - 1
        hwinc = sum(dW(R*(j-1) + 1:R*j));       % Rescaled Brownian increment.
        hdelint = hdelint + hsoln(j)*hwinc;     %  Iterative sum finaly gives
    end                                         %       the stochastic delay
                                                %                   integral.
    hsoln(i+1) = hsoln(i) + (gam*V + (al/Tdel)*hdelint^2 - (al + gam)*hsoln(i))*H; % E-M iterations.
    hsolnmat(iter,i+1) = hsoln(i+1);
end

end

%%%%%% end of Euler - Maruyama Iterations %%%%%%

%%%%%%%%%%%%%% end of programme %%%%%%%%%%%%%%%%
```

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